Representation Theory

Throughout the rest of these notes L will be a finite dimensional semisimple Lie algebra over $F = \mathbb{C}$ with CSA H, root system Φ, base Δ and Weyl group W. Although L will be finite dimensional, we need to consider infinite dimensional representations V of L. The main goal will be to explain the Weyl character formula. The proof will come afterwards.

20. Weights and maximal vectors

The statement is: Irreducible representations V of L are uniquely determined up to isomorphism by their highest weight and are generated by any vector of highest weight. This is true when V is finite dimensional and is also true for many infinite dimensional V. The main problem is that an infinite dimensional representation may not have a highest weight.

20.1. definitions.

Recall that L has a root space decomposition

$$L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$$

For any representation V of L and any $\lambda : H \to F = \mathbb{C}$ recall that the λ weight space of V is:

$$V_\lambda = \{v \in V \mid h(v) = \lambda(h)v\}$$

Let V' be the sum of all the weight spaces V_λ.

Proposition 20.1.1. (1)

$$V' = \bigoplus_{\lambda} V_\lambda$$

(2) $L_\alpha(V_\lambda) \subseteq V_{\lambda+\alpha}$.
(3) $V' = V$ *if* V *is finite dimensional.*

Definition 20.1.2. A *highest weight* for V is a weight λ so that $V_\lambda \neq 0$ but $V_{\lambda+\alpha} = 0$ for all $\alpha \in \Phi_+$.

It is clear that any (nonzero) finite dimensional representation has a highest weight.

Example 20.1.3. For the adjoint representation $V = L$, the highest weight is equal to the maximal root.

Definition 20.1.4. A *maximal vector* $v^+ \in V$ is a nonzero element with the property that

$$x_\alpha v^+ = 0$$

for all $x_\alpha \in L_\alpha$ where α is a positive root.

It is clear that any nonzero vector of highest weight is a maximal vector. The converse is not true.

It is enough to have $x_\alpha v^+ - 0$ for $\alpha \in \Delta$.

Example 20.1.5. Let $L = \mathfrak{sl}(2, F) = H \oplus L_\alpha \oplus L_{-\alpha}$. Recall that $H = Fh_\alpha, L_\alpha = Fx_\alpha, L_{-\alpha} = Fy_\alpha$. Since there is only one positive root α, a maximal weight in a representation V is any nonzero $v \in V$ so that $x_\alpha(v) = 0$.

Let $V = \mathfrak{sl}(3, F)$ with positive roots $\alpha, \beta, \alpha + \beta$. The weight space decomposition of V is

$$V = V_\alpha \oplus V_{\frac{1}{2}\alpha} \oplus V_0 \oplus V_{-\frac{1}{2}\alpha} \oplus V_{-\alpha}$$

Identifying $\alpha = 2$ since H^* is one dimensional and $\alpha(h_\alpha) = 2$, this can be rewritten:

$$V = V_2 \oplus V_1 \oplus V_0 \oplus V_{-1} \oplus V_{-2}$$

- The vector $x_\alpha \in V_2$ is a maximal vector since it has highest weight.
- The vector $x_{\alpha+\beta} \in V$ is maximal since $[x_\alpha, x_{\alpha+\beta}] = 0$. It also lies in V_1:

  $$h_\alpha(x_{\alpha+\beta}) = (\alpha(h_\alpha) + \beta(h_\alpha))x_{\alpha+\beta} = (2 - 1)x_{\alpha+\beta} = x_{\alpha+\beta}$$

  so it has highest weight since $V_{\frac{1}{2}\alpha+\alpha} = V_3 = 0$.
- The vector $h_\alpha + 2h_\beta \in V_0$ is also a maximal vector since

  $$x_\alpha(h_\alpha + 2h_\beta) = -[h_\alpha + 2h_\beta, x_\alpha] = -(\alpha(h_\alpha) + 2\beta(h_\alpha))x_\alpha = -(2 - 2)x_\alpha = 0$$

  but $h_\beta \in V_0$ so it does not have highest weight.

Note that, in this example, V has two highest weights.

20.2. **Standard cyclic modules.**

Definition 20.2.1. A *standard cyclic module* of highest weight λ is a representation V which is generated by a single maximal vector v^+ of weight λ.

This means that V is spanned by elements of the form $a_1 a_2 \cdots a_m v^+$ where $a_i \in L$. I.e., $V = \mathcal{U}(L)v^+$. The fact that the finite dimensional Lie algebra L can have infinite dimensional cyclic modules comes from the fact that $\mathcal{U}(L)$ is infinite dimensional in general.

Lemma 20.2.2. *Let V be a standard cyclic module generated by $v^+ \in V_\lambda$. Then V is spanned by elements of the form*

$$y_{\beta_1} y_{\beta_2} \cdots y_{\beta_k} v^+$$

where β_i are positive roots and $y_\beta \in L_{-\beta}$.

Proof. Use PBW to see that $\mathcal{U}(L)v^+ = \mathcal{U}(N_-(L))\mathcal{U}(B(\Delta))v^+ = \mathcal{U}(N_-(L))v^+$ □

Theorem 20.2.3. *If V is standard cyclic as above then*

1. *λ is a highest weight.*
2. *V_λ is one dimensional.*
3. *V has a weight space decomposition $V = \bigoplus V_\beta$ where β runs over weights of the form $\lambda - \sum k_i \alpha_i$ where $\alpha_i \in \Delta$ and k_i are nonnegative integers.*

In the proof of the corollary below we used the following lemma.

Lemma 20.2.4. *$v^+ \in V$ is a maximal vector iff it satisfies the condition:*

$$Bv^+ = \mathbb{C}v^+$$

In other words, v^+ is a common eigenvector for all elements of the Borel subalgebra $B = B(\Delta)$.

Proof. Let $W = \mathbb{C}v^+$. Then W is a representation of B and therefore also of $H \subseteq B$. So, v^+ is an eigenvector of H and we have a linear map $\lambda : H \to \mathbb{C}$ given by $\lambda(h)v^+ = h(v^+)$. Thus $W = W_\lambda$. For any positive root α we have $x_\alpha \in B$ and $x_a(v^+) \subseteq W_{\lambda+\alpha} = 0$. So, v^+ is a maximal vector of weight λ. The converse is obvious. □

Corollary 20.2.5. *V is indecomposable and all quotient modules are cyclic with highest weight λ. V has a unique maximal proper submodule. If V is irreducible then λ is unique.*

Proof. Suppose that $V = V_1 \oplus V_2$. Then each element of V has two coordinates. So, $v^+ = (v_1^+, v_2^+)$. For every $b \in B$ we have $bv^+ = av^+$ for some $a \in \mathbb{C}$. But $av^+ = (av_1^+, av_2^+)$. So, $Bv_1^+ = \mathbb{C}v_1^+$ and $Bv_2^+ = \mathbb{C}v_2^+$. Therefore, $(v_1^+, 0)$ and $(0, v_2^+)$ are maximal vectors of weight λ. But V_λ is one-dimensional. So, either $v_1^+ = 0$ or $v_2^+ = 0$. Since v^+ generates V, v_i^+ generates V_i. So, either $V_1 = 0$ or $V_2 = 0$ showing that V is indecomposable.

Given any submodule W of V, since W is an H-submodule of V, it must be the sum of weight spaces W_μ. Since $W \neq V$, we must have $W_\lambda = 0$. So, $(V/W)_\lambda = V_\lambda/W_\lambda = V_\lambda \neq 0$. So, $v^+ + W$ is a nonzero maximal vector for V/W of weight λ and it clearly generates V/W. So, V/W is cyclic.

To show that there is a unique maximal proper submodule, note that all proper submodules of V lie in the vector subspace $\bigoplus_{\mu\neq\lambda} V_\mu$. But then the sum of all proper submodules of V is a proper submodule which is unique since it contains all other proper submodules.

Finally, if V is irreducible then λ is uniquely determined since, given any other maximal vector $w^+ \in V_\mu$, the submodule generated by w^+ must be equal to V. But then $\lambda = \mu - \sum k_i\alpha_i$ which implies that $\mu = \lambda + \sum k_i\alpha_i$ which implies that $\lambda = \mu$. □

20.3. **Existence and uniqueness of cyclic modules.** I proved the existence theorem first:

Theorem 20.3.1. *For any $\lambda : H \to \mathbb{C}$, there exists an irreducible standard cyclic module with highest weight λ.*

Proof. Start with a one dimensional representation $D_\lambda = \mathbb{C}v^+$ of B given by taking the action of any $h \in H$ to be multiplication by $\lambda(h)$ and the action of any $x_\alpha \in L_\alpha$ to be zero. Then take:

$$V = \mathcal{U}(L) \otimes_{\mathcal{U}(B)} D_\lambda$$

This is the L-module obtained from D_λ by "extension of scalars" which is also called the "induced representation." (Recall that for any homomorphism of rings $R \to S$ and any S-module M we have an R-module given by $R \otimes_S M$.)

The L-module V is generated by the element $1 \otimes v^+$ which is a maximal vector of weight λ since $b(1 \otimes v^+) = 1 \otimes bv^+$ is a scalar multiple of $1 \otimes v^+$ and that scalar is equal to $\lambda(h)$ when $b = h \in H$.

By the corollary, V has a unique maximal proper submodule M and the quotient V/M is the desired irreducible cyclic module with prescribed highest weight λ. □

Theorem 20.3.2. *There is only one irreducible V with highest weight λ (up to isomorphism).*

Proof. Suppose there are two of irreducible standard cyclic modules V^1, V^2 with the same highest weight λ. Then $V^1_\lambda = \mathbb{C}v_1$ and $V^2_\lambda = \mathbb{C}v_2$. Let $V = V^1 \oplus V^2$. Then $V_\lambda = V^1_\lambda \oplus V^2_\lambda$. So, $v^+ = (v_1^+, v_2^+)$ is a maximal vector since, for all $b \in B$ we have $bv^+ = (bv_1^+, bv_2^+) = (av_1^+, av_2^+) = av^+$ for some scalar a. (Since a is uniquely determined by b and λ, it is the same scalar in both coordinates.)

Let W be the cyclic module generated by v^+. Then the projection map $p_1 : W \to V_1$ is onto since it sends v^+ to the generator v_1^+ of V_1. Since V_1 is irreducible, the kernel of p_1 is the unique maximal proper submodule M of W. So, $V_1 \cong W/M$. Similarly, $V_2 \cong W/M$. So, $V_1 \cong V_2$. Furthermore, this isomorphism sends v_1^+ to v_2^+. □