Representation Theory

Throughout the rest of these notes $L$ will be a finite dimensional semisimple Lie algebra over $F = \mathbb{C}$ with CSA $H$, root system $\Phi$, base $\Delta$ and Weyl group $W$. Although $L$ will be finite dimensional, we need to consider infinite dimensional representations $V$ of $L$. The main goal will be to explain the Weyl character formula. The proof will come afterwards.

20. Weights and maximal vectors

The statement is: Irreducible representations $V$ of $L$ are uniquely determined up to isomorphism by their highest weight and are generated by any vector of highest weight. This is true when $V$ is finite dimensional and is also true for many infinite dimensional $V$. The main problem is that an infinite dimensional representation may not have a highest weight.

20.1. Definitions. Recall that $L$ has a root space decomposition

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_{\alpha}$$

For any representation $V$ of $L$ and any $\lambda : H \rightarrow F = \mathbb{C}$ recall that the $\lambda$ weight space of $V$ is:

$$V_{\lambda} = \{ v \in V \mid h(v) = \lambda(h)v \}$$

Let $V'$ be the sum of all the weight spaces $V_{\lambda}$.

Proposition 20.1.1. (1)

$$V' = \bigoplus_{\lambda} V_{\lambda}$$

(2) $L_{\alpha}(V_{\lambda}) \subseteq V_{\lambda+\alpha}$.

(3) $V' = V$ if $V$ is finite dimensional.

Definition 20.1.2. A highest weight for $V$ is a weight $\lambda$ so that $V_{\lambda} \neq 0$ but $V_{\lambda+\alpha} = 0$ for all $\alpha \in \Phi_{+}$.

It is clear that any (nonzero) finite dimensional representation has a highest weight.

Example 20.1.3. For the adjoint representation $V = L$, the highest weight is equal to the maximal root.

Definition 20.1.4. A maximal vector $v^+ \in V$ is a nonzero element with the property that

$$x_{\alpha} v^+ = 0$$

for all $x_{\alpha} \in L_{\alpha}$ where $\alpha$ is a positive root.

It is clear that any nonzero vector of highest weight is a maximal vector. The converse is not true.

It is enough to have $x_{\alpha} v^+ - 0$ for $\alpha \in \Delta$. 
Example 20.1.5. Let $L = \mathfrak{sl}(2, F) = H \oplus L_\alpha \oplus L_{-\alpha}$. Recall that $H = Fh_\alpha, L_\alpha = Fx_\alpha, L_{-\alpha} = Fy_\alpha$. Since there is only one positive root $\alpha$, a maximal weight in a representation $V$ is any nonzero $v \in V$ so that $x_\alpha(v) = 0$.

Let $V = \mathfrak{sl}(3, F)$ with positive roots $\alpha, \beta, \alpha + \beta$. The weight space decomposition of $V$ is

$$V = V_\alpha \oplus V_{\frac{1}{2} \alpha} \oplus V_0 \oplus V_{-\frac{1}{2} \alpha} \oplus V_{-\alpha}$$

Identifying $\alpha = 2$ since $H^*$ is one dimensional and $\alpha(h_\alpha) = 2$, this can be rewritten:

$$V = V_2 \oplus V_1 \oplus V_0 \oplus V_{-1} \oplus V_{-2}$$

- The vector $x_\alpha \in V_2$ is a maximal vector since it has highest weight.
- The vector $x_{\alpha + \beta} \in V$ is maximal since $[x_\alpha, x_{\alpha + \beta}] = 0$. It also lies in $V_1$:
  $$h_\alpha(x_{\alpha + \beta}) = (\alpha(h_\alpha) + \beta(h_\alpha))x_{\alpha + \beta} = (2 - 1)x_{\alpha + \beta} = x_{\alpha + \beta}$$
  so it has highest weight since $V_{\frac{1}{2} \alpha + \alpha} = V_3 = 0$.
- The vector $h_\alpha + 2h_\beta \in V_0$ is also a maximal vector since
  $$x_\alpha(h_\alpha + 2h_\beta, x_\alpha) = -(\alpha(h_\alpha) + 2\beta(h_\alpha))x_\alpha = -(2 - 2)x_\alpha = 0$$
  but $h_\beta \in V_0$ so it does not have highest weight.

Note that, in this example, $V$ has two highest weights.

20.2. Standard cyclic modules.

Definition 20.2.1. A standard cyclic module of highest weight $\lambda$ is a representation $V$ which is generated by a single maximal vector $v^+$ of weight $\lambda$.

This means that $V$ is spanned by elements of the form $a_1a_2\cdots a_nv^+$ where $a_i \in L$. I.e., $V = \mathcal{U}(L)v^+$. The fact that the finite dimensional Lie algebra $L$ can have infinite dimensional cyclic modules comes from the fact that $\mathcal{U}(L)$ is infinite dimensional in general.

Lemma 20.2.2. Let $V$ be a standard cyclic module generated by $v^+ \in V_\lambda$. Then $V$ is spanned by elements of the form

$$y_{\beta_1}y_{\beta_2}\cdots y_{\beta_k}v^+$$

where $\beta_i$ are positive roots and $y_\beta \in L_{-\beta}$.

Proof. Use PBW to see that $\mathcal{U}(L)v^+ = \mathcal{U}(N_-(L))\mathcal{U}(B(\Delta))v^+ = \mathcal{U}(N_-(L))v^+ \quad \square$

Theorem 20.2.3. If $V$ is standard cyclic as above then

1. $\lambda$ is a highest weight.
2. $V_\lambda$ is one dimensional.
3. $V$ has a weight space decomposition $V = \bigoplus V_\beta$ where $\beta$ runs over weights of the form $\lambda - \sum k_i\alpha_i$ where $\alpha_i \in \Delta$ and $k_i$ are nonnegative integers.

In the proof of the corollary below we used the following lemma.
Lemma 20.2.4. \( v^+ \in V \) is a maximal vector iff it satisfies the condition:

\[
Bv^+ = \mathbb{C}v^+
\]

In other words, \( v^+ \) is a common eigenvector for all elements of the Borel subalgebra \( B = B(\Delta) \).

Proof. Let \( W = \mathbb{C}v^+ \). Then \( W \) is a representation of \( B \) and therefore also of \( H \subseteq B \). So, \( v^+ \) is an eigenvector of \( H \) and we have a linear map \( \lambda : H \to \mathbb{C} \) given by \( \lambda(h)v^+ = h(v^+) \). Thus \( W = W_\lambda \). For any positive root \( \alpha \) we have \( x_\alpha \in B \) and \( x_\alpha(v^+) \subseteq W_{\lambda+\alpha} = 0 \). So, \( v^+ \) is a maximal vector of weight \( \lambda \). The converse is obvious. \( \Box \)

Corollary 20.2.5. \( V \) is indecomposable and all quotient modules are cyclic with highest weight \( \lambda \). \( V \) has a unique maximal proper submodule. If \( V \) is irreducible then \( \lambda \) is unique.

Proof. Suppose that \( V = V_1 \oplus V_2 \). Then each element of \( V \) has two coordinates. So, \( v^+ = (v_1^+, v_2^+) \). For every \( b \in B \) we have \( bv^+ = av^+ \) for some \( a \in \mathbb{C} \). But \( av^+ = (av_1^+, av_2^+) \). So, \( Bv_1^+ = \mathbb{C}v_1^+ \) and \( Bv_2^+ = \mathbb{C}v_2^+ \). Therefore, \( (v_1^+, 0) \) and \( (0, v_2^+) \) are maximal vectors of weight \( \lambda \). But \( V_1 \) is one-dimensional. So, either \( v_1^+ = 0 \) or \( v_2^+ = 0 \). Since \( v^+ \) generates \( V \), \( v_i^+ \) generates \( V_i \). So, either \( V_1 = 0 \) or \( V_2 = 0 \) showing that \( V \) is indecomposable.

Given any submodule \( W \) of \( V \), since \( W \) is an \( H \)-submodule of \( V \), it must be the sum of weight spaces \( W_\mu \). Since \( W \neq V \), we must have \( W_\lambda = 0 \). So, \( (V/W)_\lambda = V_\lambda/W_\lambda = V_\lambda \neq 0 \). So, \( v^+ + W \) is a nonzero maximal vector for \( V/W \) of weight \( \lambda \) and it clearly generates \( V/W \). So, \( V/W \) is cyclic.

To show that there is a unique maximal proper submodule, note that all proper submodules of \( V \) lie in the vector subspace \( \bigoplus_{\mu \neq \lambda} V_\mu \). But then the sum of all proper submodules of \( V \) is a proper submodule which is unique since it contains all other proper submodules.

Finally, if \( V \) is irreducible then \( \lambda \) is uniquely determined since, given any other maximal vector \( w^+ \in V_\mu \), the submodule generated by \( w^+ \) must be equal to \( V \). But then \( \lambda = \mu - \sum k_i \alpha_i \) which implies that \( \mu = \lambda + \sum k_i \alpha_i \) which implies that \( \lambda = \mu \). \( \Box \)

20.3. Existence and uniqueness of cyclic modules. I proved the existence theorem first:

Theorem 20.3.1. For any \( \lambda : H \to \mathbb{C} \), there exists an irreducible standard cyclic module with highest weight \( \lambda \).

Proof. Start with a one dimensional representation \( D_\lambda = \mathbb{C}v^+ \) of \( B \) given by taking the action of any \( h \in H \) to be multiplication by \( \lambda(h) \) and the action of any \( x_\alpha \in L_\alpha \) to be zero. Then take:

\[
V = U(L) \otimes_{U(B)} D_\lambda
\]

This is the \( L \)-module obtained from \( D_\lambda \) by “extension of scalars” which is also called the “induced representation.” (Recall that for any homomorphism of rings \( R \to S \) and any \( S \)-module \( M \) we have an \( R \)-module given by \( R \otimes_S M \).)

The \( L \)-module \( V \) is generated by the element \( 1 \otimes v^+ \) which is a maximal vector of weight \( \lambda \) since \( b(1 \otimes v^+) = 1 \otimes bv^+ \) is a scalar multiple of \( 1 \otimes v^+ \) and that scalar is equal to \( \lambda(h) \) when \( b = h \in H \).

By the corollary, \( V \) has a unique maximal proper submodule \( M \) and the quotient \( V/M \) is the desired irreducible cyclic module with prescribed highest weight \( \lambda \). \( \Box \)
Theorem 20.3.2. There is only one irreducible $V$ with highest weight $\lambda$ (up to isomorphism).

Proof. Suppose there are two of irreducible standard cyclic modules $V^1, V^2$ with the same highest weight $\lambda$. Then $V^1_\lambda = \mathbb{C}v_1$ and $V^2_\lambda = \mathbb{C}v_2$. Let $V = V^1 \oplus V^2$. Then $V^1_\lambda = V^1_\lambda \oplus V^2_\lambda$. So, $v^+ = (v^+_1, v^+_2)$ is a maximal vector since, for all $b \in B$ we have $bv^+ = (bv^+_1, bv^+_2) = (av^+_1, av^+_2) = av^+$ for some scalar $a$. (Since $a$ is uniquely determined by $b$ and $\lambda$, it is the same scalar in both coordinates.)

Let $W$ be the cyclic module generated by $v^+$. Then the projection map $p_1 : W \to V_1$ is onto since it sends $v^+$ to the generator $v^+_1$ of $V_1$. Since $V_1$ is irreducible, the kernel of $p_1$ is the unique maximal proper submodule $M$ of $W$. So, $V_1 \cong W/M$. Similarly, $V_2 \cong W/M$. So, $V_1 \cong V_2$. Furthermore, this isomorphism sends $v^+_1$ to $v^+_2$. \qed