Representation Theory

Throughout the rest of these notes L will be a finite dimensional semisimple Lie algebra over $F = \mathbb{C}$ with CSA H, root system Φ, base Δ and Weyl group W. Although L will be finite dimensional, we need to consider infinite dimensional representations V of L. The main goal will be to explain the Weyl character formula. The proof will come afterwards.

20. Weights and maximal vectors

The statement is: Irreducible representations V of L are uniquely determined up to isomorphism by their highest weight and are generated by any vector of highest weight. This is true when V is finite dimensional and is also true for many infinite dimensional V. The main problem is that an infinite dimensional representation may not have a highest weight.

20.1. definitions.

Recall that L has a root space decomposition

$$L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$$

For any representation V of L and any $\lambda : H \to F = \mathbb{C}$ recall that the λ weight space of V is:

$$V_\lambda = \{v \in V \mid h(v) = \lambda(h)v\}$$

Let V' be the sum of all the weight spaces V_λ.

Proposition 20.1.1. (1)

$$V' = \bigoplus_{\lambda} V_\lambda$$

(2) $L_\alpha(V_\lambda) \subseteq V_{\lambda+\alpha}$.
(3) $V' = V$ *if* V *is finite dimensional.*

Definition 20.1.2. A *highest weight* for V is a weight λ so that $V_\lambda \neq 0$ but $V_{\lambda+\alpha} = 0$ for all $\alpha \in \Phi_+$.

It is clear that any (nonzero) finite dimensional representation has a highest weight.

Example 20.1.3. For the adjoint representation $V = L$, the highest weight is equal to the maximal root.

Definition 20.1.4. A *maximal vector* $v^+ \in V$ is a nonzero element with the property that

$$x_\alpha v^+ = 0$$

for all $x_\alpha \in L_\alpha$ where α is a positive root.

It is clear that any nonzero vector of highest weight is a maximal vector. The converse is not true.

It is enough to have $x_\alpha v^+ - 0$ for $\alpha \in \Delta$.

Example 20.1.5. Let $L = \mathfrak{sl}(2, F) = H \oplus L_\alpha \oplus L_{-\alpha}$. Recall that $H = Fh_\alpha, L_\alpha = Fx_\alpha, L_{-\alpha} = Fy_\alpha$. Since there is only one positive root α, a maximal weight in a representation V is any nonzero $v \in V$ so that $x_\alpha(v) = 0$.

Let $V = \mathfrak{sl}(3, F)$ with positive roots $\alpha, \beta, \alpha+\beta$. The weight space decomposition of V is

$$V = V_\alpha \oplus V_{\frac{1}{2}\alpha} \oplus V_0 \oplus V_{-\frac{1}{2}\alpha} \oplus V_{-\alpha}$$

Identifying $\alpha = 2$ since H^* is one dimensional and $\alpha(h_\alpha) = 2$, this can be rewritten:

$$V = V_2 \oplus V_1 \oplus V_0 \oplus V_{-1} \oplus V_{-2}$$

- The vector $x_\alpha \in V_2$ is a maximal vector since it has highest weight.
- The vector $x_{\alpha+\beta} \in V$ is maximal since $[x_\alpha, x_{\alpha+\beta}] = 0$. It also lies in V_1:
  $$h_\alpha(x_{\alpha+\beta}) = (\alpha(h_\alpha) + \beta(h_\alpha))x_{\alpha+\beta} = (2-1)x_{\alpha+\beta} = x_{\alpha+\beta}$$
  so it has highest weight since $V_{\frac{1}{2}\alpha+\alpha} = V_3 = 0$.
- The vector $h_\alpha + 2h_\beta \in V_0$ is also a maximal vector since
  $$x_\alpha(h_\alpha + 2h_\beta) = -[h_\alpha + 2h_\beta, x_\alpha] = -(\alpha(h_\alpha) + 2\beta(h_\alpha))x_\alpha = -(2-2)x_\alpha = 0$$
  but $h_\beta \in V_0$ so it does not have highest weight.

Note that, in this example, V has two highest weights.

20.2. **Standard cyclic modules.**

Definition 20.2.1. A *standard cyclic module* of highest weight λ is a representation V which is generated by a single maximal vector v^+ of weight λ.

This means that V is spanned by elements of the form $a_1 a_2 \cdots a_m v^+$ where $a_i \in L$. I.e., $V = \mathcal{U}(L)v^+$. The fact that the finite dimensional Lie algebra L can have infinite dimensional cyclic modules comes from the fact that $\mathcal{U}(L)$ is infinite dimensional in general.

Lemma 20.2.2. *Let V be a standard cyclic module generated by $v^+ \in V_\lambda$. Then V is spanned by elements of the form*

$$y_{\beta_1} y_{\beta_2} \cdots y_{\beta_k} v^+$$

where β_i are positive roots and $y_\beta \in L_{-\beta}$.

Proof. Use PBW to see that $\mathcal{U}(L) = \mathcal{U}(N_-(L))\mathcal{U}(B(\Delta))v^+ = \mathcal{U}(N_-(L))v^+$ □

Theorem 20.2.3. *If V is standard cyclic as above then*

1. *λ is a highest weight*
2. *V_λ is one dimensional*
3. *V has a weight space decomposition $V = \bigoplus V_\beta$ where β runs over weights of the form $\lambda - \sum k_i \alpha_i$ where $\alpha_i \in \Delta$ and k_i are nonnegative integers.*

Corollary 20.2.4. *V is indecomposable and all quotient modules are cyclic with highest weight λ. If V is irreducible then λ is unique.*

Theorem 20.2.5. *There is only one irreducible V with highest weight λ (up to isomorphism).*