21. Finite dimensional modules

Given any weight $\lambda : H \to \mathbb{C}$ we have the cyclic module

$$Z(\lambda) = \mathcal{U}(L) \otimes_{\mathcal{U}(B)} D_\lambda$$

This is called the *Verma module* of highest weight λ. This module has a unique maximal proper submodule and the quotient $V(\lambda)$ is the unique irreducible module with highest weight λ. In this section we determine when $V(\lambda)$ is finite dimensional. If we recall Weyl's Theorem (Every finite dimensional representation of a semisimple Lie algebra is a direct sum of irreducible modules.) this will give a complete classification of all finite dimensional representations of semisimple Lie algebras.

The statement is:

Theorem 21.0.3. *$V(\lambda)$ is finite dimensional iff λ is a dominant weight, i.e., $\lambda(h_\alpha) = \langle \lambda, \alpha \rangle$ is a nonnegative integer for all positive roots α.*

The theory of dominant weights was done abstractly in Section 13 which we skipped. Now we need some of the basic concepts from that section.

21.1. Dominant weights.

Suppose that $\alpha_1, \cdots, \alpha_n$ are the positive simple roots (the elements of the base Δ). For each i we have a copy $S_i = S_{\alpha_i}$ of $\mathfrak{sl}(2, F)$ with basis $x_i, y_i, h_i = h_{\alpha_i} \in H$. Then the h_i form a vector space basis for H.

Definition 21.1.1. An *abstract or integral weight* is a linear function $\lambda : H \to \mathbb{C}$ with the property that $\lambda(h_i) \in \mathbb{Z}$ for all i. An abstract weight is called *dominant* if $\lambda(h_i) \geq 0$ for all i. The *fundamental dominant weights* λ_i are the ones given by:

$$\lambda_i(h_j) = \delta_{ij}$$

I.e., these form the dual basis for the basis of H given by the h_i.

Example 21.1.2. For $L = \mathfrak{sl}(2, \mathbb{C})$ there is only one fundamental weight $\lambda_1 = \frac{1}{2}\alpha$:

$$\lambda_1(h_1) = \frac{1}{2}\alpha(h_\alpha) = \frac{2}{2} = 1$$

Exercise 21.1.3. Show that for $L = \mathfrak{sl}(3, \mathbb{C})$, the fundamental weights are

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \qquad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$

It is clear that all dominant weights are given by addition of the fundamental dominant weights:

$$\lambda = \sum n_i \lambda_i$$

where n_i are nonnegative integers.

The set of dominant weights is denoted Λ^+. A weight $\lambda = \sum n_i \lambda_i$ is called *strongly dominant* if $n_i > 0$ for all i. One important example is the minimal strongly dominant weight given by

$$\delta = \sum \lambda_i$$

This is characterized in several ways:

(1) $\delta(h_i) = 1$ for all i.
(2)
$$\delta = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$$

To prove the last equation we use the action of the Weyl group W. Let $\mu = \frac{1}{2}\sum\alpha$. Apply the simple reflection s_i given by
$$s_i(x) = x - \langle x, \alpha_i\rangle\,\alpha_i$$
We know that s_i sends α_i to $-\alpha_i$ and permutes the other positive roots. So:
$$s_i(\mu) = \mu - \alpha_i = \mu - \langle\mu, \alpha_i\rangle\,\alpha_i$$
Therefore, $\langle\mu, \alpha_i\rangle = \mu(h_i) = 1$ for all i. So, $\mu = \delta$.

21.2. **finite irreducible modules.**

Theorem 21.2.1. *If $V(\lambda)$ is finite dimensional then*

1. *λ is a dominant weight.*
2. *All weights μ of $V(\lambda)$ are integral weights and therefore given as integer linear combinations of the fundamental weights:*
$$\mu = \sum n_i\lambda_i$$
3. *The set Π of weights μ which occur in $V(\lambda)$ is saturated (defined below).*

A set Π of integral weight $\mu = \sum n_i\lambda_i$ is *saturated* if, for all $\beta\in\Phi$ and all integers $0\le m\le\langle\mu,\beta\rangle$, $\mu - m\beta\in\Pi$. Since every root is a sum of simple roots, it is enough to have this for $\beta = \alpha_i$ in which case $0\le m\le n_i = \langle\mu,\alpha_i\rangle$.

Proof. We view $V(\lambda)$ as a representation of S_i and quote results from Section 7. Each weight of $V(\lambda)_\mu$ becomes $\mu(h_i) = n_i$. Therefore, n_i must be an integer. Thus all weights μ are integral. For the highest weight λ, $\lambda(h_i)$ must be a nonnegative integer. So, λ is dominant.

The action of y_i on $V(\lambda)$ sends $V(\lambda)_\mu$ to $V(\lambda)_{\mu-\alpha_i}$ and, by symmetry of the weights of representations of S_i around 0,
$$y_i^{n_i}(w) \neq 0 \in V(\lambda)_{\mu - n_i\alpha_i}$$
for all $w \neq 0 \in V(\lambda)_\mu$. So, Π is saturated. □

Theorem 21.2.2. *If λ is any dominant weight then $V(\lambda)$ is finite dimensional. Furthermore, the set Π of weights μ is invariant under the action of the Weyl group and is minimal W-invariant saturated set of integral weights which contains λ.*

Proof. You can read the proof in the book. Here is an outline.

1. For each i the sequence of elements
$$v^+, y_iv^+, y_i^2v^+, \cdots, y_i^kv^+$$
for $k = \lambda(h_i)$ forms a finite dimensional S_i submodule of $V(\lambda)$.

(2) Let V' be the sum of all finite dimensional S_i submodules of $V(\lambda)$. Then V' is a a nonzero L-submodule and therefore $V' = V$.
(3) Recall that $\tau_i = \exp(x_i)\exp(-y_i)\exp(x_i)$ is an automorphism of $V(\lambda)$ which lifts the action of the simple reflection σ_i. Thus $\tau_i V(\lambda)_\mu = V(\lambda)_{\sigma_i\mu}$. (Since $V = V'$, $\tau_i V = \tau_i V' = V' = V$.)
(4) All weights are integral by the Key Lemma 20.2.2 we proved last time. Also $V(\lambda)_\mu$ is finite dimensional for all μ.
(5) The symmetry of Π under the Weyl group forces it to be finite.
(6) If Π' is the minimal W-invariant saturated subset of Π then $\bigoplus_{\mu\in\Pi'} V(\lambda)_\mu$ is a submodule of $V(\lambda)$ and therefore the whole thing.

□

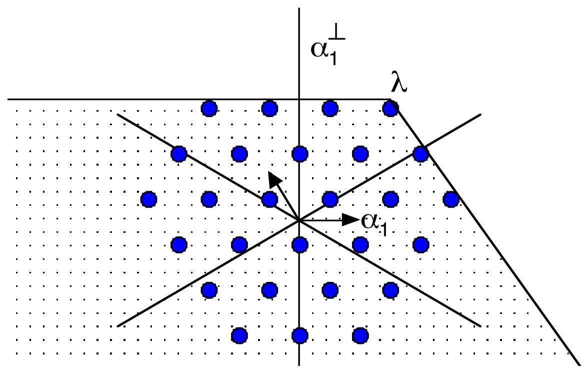

FIGURE 2. Since λ is the highest weight, Π is confined to the shaded region. Since Π is W invariant it is confined to the blue dots.

I pointed out at the end of the class that, in order to prove that $V(\lambda)$ is finite dimensional it suffices to construct a finite dimensional cyclic module of highest weight λ for any dominant weight λ since $V(\lambda)$ is uniquely determined by λ.

Lemma 21.2.3. *$V(\lambda) \otimes V(\mu)$ contains a maximal vector of highest weight $\lambda + \mu$.*

Proof. Let $v = v_1^+ \otimes v_2^+ \in V(\lambda) \otimes V(\mu)$. Then, for any $h \in H$ we have

$$h(v) = h(v_1^+) \otimes v_2^+ + v_1^+ \otimes h(v_2^+) = \lambda(h)v_1^+ \otimes v_2^+ + v_1^+ \otimes \mu(h)v_2^+ = (\lambda + \mu)(h)v_1^+ \otimes v_2^+$$

Therefore $v = v_1^+ \otimes v_2^+$ has weight $\lambda + \mu$. Also, for any x_α for positive root α we have

$$x_\alpha(v_1^+ \otimes v_2^+) = x_\alpha(v_1^+) \otimes v_2^+ + v_1^+ \otimes x_\alpha(v_2^+) = 0 + 0 = 0$$

So, $v = v_1^+ \otimes v_2^+$ is a maximal vector. □

This elementary lemma implies that, to prove Theorem 21.2.2 it suffices to construct a finite dimensional module containing $V(\lambda_i)$ for the fundamental dominant weights λ_i. We will do this later at least in some case.