21. Finite dimensional modules

Given any weight $\lambda : H \to \mathbb{C}$ we have the cyclic module

$$Z(\lambda) = U(L) \otimes_{U(B)} D_\lambda$$

This is called the Verma module of highest weight $\lambda$. This module has a unique maximal proper submodule and the quotient $V(\lambda)$ is the unique irreducible module with highest weight $\lambda$. In this section we determine when $V(\lambda)$ is finite dimensional. If we recall Weyl’s Theorem (Every finite dimensional representation of a semisimple Lie algebra is a direct sum of irreducible modules.) this will give a complete classification of all finite dimensional representations of semisimple Lie algebras.

The statement is:

**Theorem 21.0.3.** $V(\lambda)$ is finite dimensional iff $\lambda$ is a dominant weight, i.e., $\lambda(h_\alpha) = \langle \lambda, \alpha \rangle$ is a nonnegative integer for all positive roots $\alpha$.

The theory of dominant weights was done abstractly in Section 13 which we skipped. Now we need some of the basic concepts from that section.

21.1. Dominant weights. Suppose that $\alpha_1, \ldots, \alpha_n$ are the positive simple roots (the elements of the base $\Delta$). For each $i$ we have a copy $S_i = S_{\alpha_i}$ of $\mathfrak{sl}(2, F)$ with basis $x_i, y_i, h_i = h_{\alpha_i} \in H$. Then the $h_i$ form a vector space basis for $H$.

**Definition 21.1.1.** An abstract or integral weight is a linear function $\lambda : H \to \mathbb{C}$ with the property that $\lambda(h_i) \in \mathbb{Z}$ for all $i$. An abstract weight is called dominant if $\lambda(h_i) \geq 0$ for all $i$. The fundamental dominant weights $\lambda_i$ are the ones given by:

$$\lambda_i(h_j) = \delta_{ij}$$

I.e., these form the dual basis for the basis of $H$ given by the $h_i$.

**Example 21.1.2.** For $L = \mathfrak{sl}(2, \mathbb{C})$ there is only one fundamental weight $\lambda_1 = \frac{1}{2} \alpha$:

$$\lambda_1(h_1) = \frac{1}{2} \alpha(h_\alpha) = \frac{2}{2} = 1$$

**Exercise 21.1.3.** Show that for $L = \mathfrak{sl}(3, \mathbb{C})$, the fundamental weights are

$$\lambda_1 = \frac{2}{3} \alpha_1 + \frac{1}{3} \alpha_2, \quad \lambda_2 = \frac{1}{3} \alpha_1 + \frac{2}{3} \alpha_2$$

It is clear that all dominant weights are given by addition of the fundamental dominant weights:

$$\lambda = \sum n_i \lambda_i$$

where $n_i$ are nonnegative integers.

The set of dominant weights is denoted $\Lambda^+$. A weight $\lambda = \sum n_i \lambda_i$ is called strongly dominant if $n_i > 0$ for all $i$. One important example is the minimal strongly dominant weight given by

$$\delta = \sum \lambda_i$$

This is characterized in several ways:
(1) $\delta(h_i) = 1$ for all $i$.

(2)

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$$

To prove the last equation we use the action of the Weyl group $W$. Let $\mu = \frac{1}{2} \sum \alpha$. Apply the simple reflection $s_i$ given by

$$s_i(x) = x - \langle x, \alpha_i \rangle \alpha_i$$

We know that $s_i$ sends $\alpha_i$ to $-\alpha_i$ and permutes the other positive roots. So:

$$s_i(\mu) = \mu - \alpha_i = \mu - \langle \mu, \alpha_i \rangle \alpha_i$$

Therefore, $\langle \mu, \alpha_i \rangle = \mu(h_i) = 1$ for all $i$. So, $\mu = \delta$.

21.2. finite irreducible modules.

**Theorem 21.2.1.** If $V(\lambda)$ is finite dimensional then

1. $\lambda$ is a dominant weight.
2. All weights $\mu$ of $V(\lambda)$ are integral weights and therefore given as integer linear combinations of the fundamental weights:

$$\mu = \sum n_i \lambda_i$$

3. The set $\Pi$ of weights $\mu$ which occur in $V(\lambda)$ is saturated (defined below).

A set $\Pi$ of integral weight $\mu = \sum n_i \lambda_i$ is saturated if, for all $\beta \in \Phi$ and all integers $0 \leq m \leq \langle \mu, \beta \rangle$, $\mu - m \beta \in \Pi$. Since every root is a sum of simple roots, it is enough to have this for $\beta = \alpha_i$ in which case $0 \leq m \leq n_i = \langle \mu, \alpha_i \rangle$.

**Proof.** We view $V(\lambda)$ as a representation of $S_i$ and quote results from Section 7. Each weight of $V(\lambda)_\mu$ becomes $\mu(h_i) = n_i$. Therefore, $n_i$ must be an integer. Thus all weights $\mu$ are integral. For the highest weight $\lambda$, $\lambda(h_i)$ must be a nonnegative integer. So, $\lambda$ is dominant.

The action of $y_i$ on $V(\lambda)$ sends $V(\lambda)_\mu$ to $V(\lambda)_{\mu-\alpha_i}$ and, by symmetry of the weights of representations of $S_i$ around 0,

$$y_i^n(w) \neq 0 \in V(\lambda)_{\mu-n\alpha_i}$$

for all $w \neq 0 \in V(\lambda)_\mu$. So, $\Pi$ is saturated.

**Theorem 21.2.2.** If $\lambda$ is any dominant weight then $V(\lambda)$ is finite dimensional. Furthermore, the set $\Pi$ of weights $\mu$ is invariant under the action of the Weyl group and is minimal $W$-invariant saturated set of integral weights which contains $\lambda$.

**Proof.** You can read the proof in the book. Here is an outline.

1. For each $i$ the sequence of elements

$$v^+, y_i v^+, y_i^2 v^+, \cdots, y_i^k v^+$$

for $k = \lambda(h_i)$ forms a finite dimensional $S_i$ submodule of $V(\lambda)$. 

(2) Let $V'$ be the sum of all finite dimensional $S_i$ submodules of $V(\lambda)$. Then $V'$ is a nonzero $L$-submodule and therefore $V' = V$.

(3) Recall that $\tau_i = \exp(x_i)\exp(-y_i)\exp(x_i)$ is an automorphism of $V(\lambda)$ which lifts the action of the simple reflection $\sigma_i$. Thus $\tau_i V(\lambda)_{\mu} = V(\lambda)_{\sigma_i \mu}$. (Since $V = V'$, $\tau_i V = \tau_i V' = V' = V$.)

(4) All weights are integral by the Key Lemma 20.2.2 we proved last time. Also $V(\lambda)_{\mu}$ is finite dimensional for all $\mu$.

(5) The symmetry of $\Pi$ under the Weyl group forces it to be finite.

(6) If $\Pi'$ is the minimal $W$-invariant saturated subset of $\Pi$ then $\bigoplus_{\mu \in \Pi'} V(\lambda)_{\mu}$ is a submodule of $V(\lambda)$ and therefore the whole thing.

I pointed out at the end of the class that, in order to prove that $V(\lambda)$ is finite dimensional it suffices to construct a finite dimensional cyclic module of highest weight $\lambda$ for any dominant weight $\lambda$ since $V(\lambda)$ is uniquely determined by $\lambda$.

**Lemma 21.2.3.** $V(\lambda) \otimes V(\mu)$ contains a maximal vector of highest weight $\lambda + \mu$.

**Proof.** Let $v = v_1^+ \otimes v_2^+ \in V(\lambda) \otimes V(\mu)$. Then, for any $h \in H$ we have

$$h(v) = h(v_1^+ \otimes v_2^+) = \lambda(v_1^+ \otimes v_2^+) + \mu(h)v_2^+ = (\lambda + \mu)(h)v_1^+ \otimes v_2^+$$

Therefore $v = v_1^+ \otimes v_2^+$ has weight $\lambda + \mu$. Also, for any $x_\alpha$ for positive root $\alpha$ we have

$$x_\alpha(v_1^+ \otimes v_2^+) = x_\alpha(v_1^+) \otimes v_2^+ + v_1^+ \otimes x_\alpha(v_2^+) = 0 + 0 = 0$$

So, $v = v_1^+ \otimes v_2^+$ is a maximal vector. \qed

This elementary lemma implies that, to prove Theorem 21.2.2 it suffices to construct a finite dimensional module containing $V(\lambda_i)$ for the fundamental dominant weights $\lambda_i$. We will do this later at least in some case.