

## 21. FINITE DIMENSIONAL MODULES

Given any weight  $\lambda : H \rightarrow \mathbb{C}$  we have the cyclic module

$$Z(\lambda) = \mathcal{U}(L) \otimes_{\mathcal{U}(B)} D_\lambda$$

This is called the *Verma module* of highest weight  $\lambda$ . This module has a unique maximal proper submodule and the quotient  $V(\lambda)$  is the unique irreducible module with highest weight  $\lambda$ . In this section we determine when  $V(\lambda)$  is finite dimensional. If we recall Weyl's Theorem (Every finite dimensional representation of a semisimple Lie algebra is a direct sum of irreducible modules.) this will give a complete classification of all finite dimensional representations of semisimple Lie algebras.

The statement is:

**Theorem 21.0.3.**  $V(\lambda)$  is finite dimensional iff  $\lambda$  is a dominant weight, i.e.,  $\lambda(h_\alpha) = \langle \lambda, \alpha \rangle$  is a nonnegative integer for all positive roots  $\alpha$ .

The theory of dominant weights was done abstractly in Section 13 which we skipped. Now we need some of the basic concepts from that section.

**21.1. Dominant weights.** Suppose that  $\alpha_1, \dots, \alpha_n$  are the positive simple roots (the elements of the base  $\Delta$ ). For each  $i$  we have a copy  $S_i = S_{\alpha_i}$  of  $\mathfrak{sl}(2, F)$  with basis  $x_i, y_i, h_i = h_{\alpha_i} \in H$ . Then the  $h_i$  form a vector space basis for  $H$ .

**Definition 21.1.1.** An *abstract or integral weight* is a linear function  $\lambda : H \rightarrow \mathbb{C}$  with the property that  $\lambda(h_i) \in \mathbb{Z}$  for all  $i$ . An abstract weight is called *dominant* if  $\lambda(h_i) \geq 0$  for all  $i$ . The *fundamental dominant weights*  $\lambda_i$  are the ones given by:

$$\lambda_i(h_j) = \delta_{ij}$$

I.e., these form the dual basis for the basis of  $H$  given by the  $h_i$ .

**Example 21.1.2.** For  $L = \mathfrak{sl}(2, \mathbb{C})$  there is only one fundamental weight  $\lambda_1 = \frac{1}{2}\alpha$ :

$$\lambda_1(h_1) = \frac{1}{2}\alpha(h_\alpha) = \frac{2}{2} = 1$$

**Exercise 21.1.3.** Show that for  $L = \mathfrak{sl}(3, \mathbb{C})$ , the fundamental weights are

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$

It is clear that all dominant weights are given by addition of the fundamental dominant weights:

$$\lambda = \sum n_i \lambda_i$$

where  $n_i$  are nonnegative integers.

The set of dominant weights is denoted  $\Lambda^+$ . A weight  $\lambda = \sum n_i \lambda_i$  is called *strongly dominant* if  $n_i > 0$  for all  $i$ . One important example is the minimal strongly dominant weight given by

$$\delta = \sum \lambda_i$$

This is characterized in several ways:

- (1)  $\delta(h_i) = 1$  for all  $i$ .
- (2)

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$$

To prove the last equation we use the action of the Weyl group  $W$ . Let  $\mu = \frac{1}{2} \sum \alpha$ . Apply the simple reflection  $s_i$  given by

$$s_i(x) = x - \langle x, \alpha_i \rangle \alpha_i$$

We know that  $s_i$  sends  $\alpha_i$  to  $-\alpha_i$  and permutes the other positive roots. So:

$$s_i(\mu) = \mu - \alpha_i = \mu - \langle \mu, \alpha_i \rangle \alpha_i$$

Therefore,  $\langle \mu, \alpha_i \rangle = \mu(h_i) = 1$  for all  $i$ . So,  $\mu = \delta$ .

**21.2. finite irreducible modules.**

**Theorem 21.2.1.** *If  $V(\lambda)$  is finite dimensional then*

- (1)  $\lambda$  is a dominant weight.
- (2) All weights  $\mu$  of  $V(\lambda)$  are integral weights and therefore given as integer linear combinations of the fundamental weights:

$$\mu = \sum n_i \lambda_i$$

- (3) The set  $\Pi$  of weights  $\mu$  which occur in  $V(\lambda)$  is saturated (defined below).

A set  $\Pi$  of integral weight  $\mu = \sum n_i \lambda_i$  is *saturated* if, for all  $\beta \in \Phi$  and all integers  $0 \leq m \leq \langle \mu, \beta \rangle$ ,  $\mu - m\beta \in \Pi$ . Since every root is a sum of simple roots, it is enough to have this for  $\beta = \alpha_i$  in which case  $0 \leq m \leq n_i = \langle \mu, \alpha_i \rangle$ .

*Proof.* We view  $V(\lambda)$  as a representation of  $S_i$  and quote results from Section 7. Each weight of  $V(\lambda)_\mu$  becomes  $\mu(h_i) = n_i$ . Therefore,  $n_i$  must be an integer. Thus all weights  $\mu$  are integral. For the highest weight  $\lambda$ ,  $\lambda(h_i)$  must be a nonnegative integer. So,  $\lambda$  is dominant.

The action of  $y_i$  on  $V(\lambda)$  sends  $V(\lambda)_\mu$  to  $V(\lambda)_{\mu-\alpha_i}$  and, by symmetry of the weights of representations of  $S_i$  around 0,

$$y_i^{n_i}(w) \neq 0 \in V(\lambda)_{\mu-n_i\alpha_i}$$

for all  $w \neq 0 \in V(\lambda)_\mu$ . So,  $\Pi$  is saturated. □

**Theorem 21.2.2.** *If  $\lambda$  is any dominant weight then  $V(\lambda)$  is finite dimensional. Furthermore, the set  $\Pi$  of weights  $\mu$  is invariant under the action of the Weyl group and is minimal  $W$ -invariant saturated set of integral weights which contains  $\lambda$ .*

*Proof.* You can read the proof in the book. Here is an outline.

- (1) For each  $i$  the sequence of elements

$$v^+, y_i v^+, y_i^2 v^+, \dots, y_i^k v^+$$

for  $k = \lambda(h_i)$  forms a finite dimensional  $S_i$  submodule of  $V(\lambda)$ .

- (2) Let  $V'$  be the sum of all finite dimensional  $S_i$  submodules of  $V(\lambda)$ .

- (3) Recall that  $\tau_i = \exp(x_i)\exp(-y_i)\exp(x_i)$  is an automorphism of  $V(\lambda)$  which lifts the action of the simple reflection  $\sigma_i$ . Thus  $\tau_i V(\lambda)_\mu = V(\lambda)_{\sigma_i \mu}$ .
- (4) All weights are integral by the Key Lemma 20.2.2 we proved last time. Also  $V(\lambda)_\mu$  is finite dimensional for all  $\mu$ .
- (5) The symmetry of  $\Pi$  under the Weyl group forces it to be finite.
- (6) If  $\Pi'$  is the minimal  $W$ -invariant saturated subset of  $\Pi$  then  $\bigoplus_{\mu \in \Pi'} V(\lambda)_\mu$  is a submodule of  $V(\lambda)$  and therefore the whole thing.

□