22. Formal characters

The main purpose of this section is to set up the notation in order to state the Weyl character formula. The details of the formula, examples and proofs will be given in the remaining lectures of this course.

22.1. **Definition.** (Section 22.5 in book)
Let $\Lambda \subseteq H^*$ be the set of integral weights. This is a free abelian group generated by the fundamental dominant weights λ_i.

$$\Lambda \cong \mathbb{Z}^n$$

If λ is a dominant weight then we recall that $V(\lambda)$ is finite dimensional and the set Π of all weights μ of $V(\lambda)$ is a subset of Λ. We also recall one of the key steps in the proof of these facts:

Proposition 22.1.1. *The action of Weyl group W on H^* leaves Λ invariant and fixes the representation $V(\lambda)$ ($\sigma V(\lambda) \cong V(\lambda)$) and therefore leaves Π invariant.*

We will give several formulas for the dimension of $V(\lambda)_\mu$ which we denote

$$m_\lambda(\mu) := \dim V(\lambda)_\mu$$

Let $\mathbb{Z}[\Lambda]$ be the integer group ring of the group Λ. Additively, this is the free abelian group generated by the elements of Λ which we now need to write multiplicatively. Thus $e(\mu)$ denotes the element of $\mathbb{Z}[\Lambda]$ corresponding to $\mu \in \Lambda$ and

$$e(\lambda + \mu) = e(\lambda)e(\mu)$$

Elements of $\mathbb{Z}[\Lambda]$ are finite formal linear combinations

$$\sum n_i e(\mu_i)$$

The book also writes this as

$$\sum f(\mu)e(\mu)$$

where $f : \Lambda \to \mathbb{Z}$ is a set mapping with finite support (the *support* of a function is the subset of the domain on which it is nonzero). Thus $f(\mu) = 0$ for all but a finite number of μ.

Definition 22.1.2. For any finite dimensional representation V of a semisimple Lie algebra L we define the *(formal) character* ch_V of V to be the element of $\mathbb{Z}[\Lambda]$ given by

$$\mathrm{ch}_V = \sum_\mu \dim V_\mu e(\mu)$$

When $V = V(\lambda)$ we use the notation $\mathrm{ch}_\lambda = \mathrm{ch}_{V(\lambda)}$. Thus

$$\mathrm{ch}_\lambda = \sum_\mu m_\lambda(\mu)e(\mu)$$

22.2. **Basic properties.** The basic properties of the formal character are easy to prove.

Proposition 22.2.1. *The formal character is additive:*

$$\mathrm{ch}_{V\oplus W} = \mathrm{ch}_V + \mathrm{ch}_W$$

Proof. $(V\oplus W)_\mu = V_\mu \oplus W_\mu$. So, $\dim(V\oplus W)_\mu = \dim V_\mu + \dim W_\mu$. □

Proposition 22.2.2. *The formal character is multiplicative:*

$$\mathrm{ch}_{V\otimes W} = \mathrm{ch}_V \mathrm{ch}_W$$

Proof. As I pointed out last time:

$$V_\lambda \otimes W_\mu \subseteq (V\otimes W)_{\lambda+\mu}$$

Therefore,

$$(V\otimes W)_\lambda = \bigoplus_{\mu+\nu=\lambda} V_\mu \otimes W_\nu$$

Taking dimensions, we get:

$$m_{V\otimes W}(\lambda) = \sum_{\mu+\nu=\lambda} m_V(\mu) m_W(\nu)$$

which implies $\mathrm{ch}_{V\otimes W} = \mathrm{ch}_V \mathrm{ch}_W$ by definition of multiplication of elements of $\mathbb{Z}[\Lambda]$. □

Proposition 22.2.3. *The character of any finite dimensional representation V is fixed under the action of the Weyl group:*

$$\sigma \mathrm{ch}_V = \mathrm{ch}_{\sigma V} = \mathrm{ch}_V$$

for any $\sigma \in W$.

Proof. Since V is finite dimensional, it is a direct sum of $V(\lambda)$ where λ is dominant. But then $\sigma V = \bigoplus \sigma V(\lambda) \cong \bigoplus V(\lambda) = V$ for all $\sigma \in W$ since $\sigma V(\lambda) \cong V(\lambda)$. □

Lemma 22.2.4. *$\lambda \in \Lambda$ is dominant iff it lies in the fundamental Weyl chamber. In particular, every integral weight μ is in the W-orbit of a dominant weight.*

Proof. This follows from the definition since $\lambda(h_i) = \langle \lambda, \alpha_i \rangle$ is ≥ 0 for all i iff λ lies on the nonnegative side of the hyperplane perpendicular to each simple root α_i. □

Proposition 22.2.5. *Any element of $\mathbb{Z}[\Lambda]$ which is fixed by the action of W can be expressed uniquely as an integer linear combination of ch_λ where $\lambda \in \Lambda^+$.*

Proof. Take any element $f \in \mathbb{Z}[\Lambda]$ which is invariant under the action of W. View f as a function $f : \Lambda \to \mathbb{Z}$ with finite support. Let M_f be the set of all dominant weights μ so that $f(\lambda) \neq 0$ for a dominant weight $\lambda \geq \mu$. We will show by induction on the size of M_f that f is an integer linear combination of the ch_λ. If M_f is empty then $f = 0$ by the lemma. So, suppose that M_f is nonempty and we know the existence statement for all smaller M_f. Let λ be a maximal element of M_f. Then $f(\lambda) = a \in \mathbb{Z}, a \neq 0$. Let $g = f - a\mathrm{ch}_\lambda$ then $M_g \subset M_f$ but $\lambda \notin M_g$. Therefore, g is an integer linear combination of ch_μ's. So, $f = g + \mathrm{ch}_\lambda$.

Uniqueness is easy. Suppose that we have two expressions for f as an integer linear combination of ch_λ's. Then the difference is a linear combination which gives 0. Putting all positive terms on one side of the equation and negative terms on the other, this can be written as a equality $\mathrm{ch}_V = \mathrm{ch}_W$ between two representations V, W. Take $\lambda \in \Lambda^+$ maximal so that λ occurs as a weight of V and W. Then $V(\lambda)$ must be a direct summand of both V and W and by induction on their size we get $V \cong W$. So, the expression is unique. □

22.3. **Representation ring.** The properties of the formal character ch_V can be summarized by saying that it gives an isomorphism

$$\mathrm{ch} : Rep(L) \cong \mathbb{Z}[\Lambda]^W$$

between the representation ring of L and the ring of W-invariant elements of the integer group ring of Λ.

Definition 22.3.1. The *representation ring* $Rep(L)$ of L is defined as follows. As an additive group, $Rep(\Lambda)$ is the free additive group generated by isomorphism classes $[V]$ of finite dimensional representations V of L modulo the relation:

$$[V] + [W] = [V \oplus W]$$

The multiplication on $Rep(L)$ is given by:

$$[V] \cdot [W] = [V \otimes W]$$

The additive and multiplicative properties of ch implty that $\mathrm{ch} : Rep(L) \to \mathbb{Z}[\Lambda]$ is a ring homomorphism. (And ch sends the unit $\mathbb{C}$ of $Rep(L)$ to the unit $1 = e(0)$ of $\mathbb{Z}[\Lambda]$ implying that ch is unital.) The W-invariance property of ch_V imply that the image of ch is contained in $\mathbb{Z}[\Lambda]^W$ and the last proposition implies that ch is a monomorphism with image $\mathbb{Z}[\Lambda]$.

Example 22.3.2. In the case $L = \mathfrak{sl}(2, F)$, the irreducible representations are $V(m), m \geq 0$ with highest weight m. So, the group ring $\mathbb{Z}[\Lambda]$ is isomorphic to $\mathbb{Z}[t, t^{-1}]$ with $t^m = e(m)$. The nontrivial element of the Weyl group $\mathbb{Z}/2$ acts by $t \leftrightarrow t^{-1}$. So, the W-invariant subring consists of all linear combinations of the form:

$$a_0 + a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) + \cdots + a_n(t^n + t^{-n})$$

with multiplication of the generators $s_n = t^n + t^{-n}$ given by

$$s_n s_m = (t^n + t^{-n})(t^m + t^{-m}) = t^{n+m} + t^{n-m} + t^{m-n} + t^{-n-m} = s_{n+m} + s_{|n-m|}$$

The representation ring of $\mathfrak{sl}(2, F)$ is the additive group of all linear combinations

$$b_0[V(0)] + b_1[V(1)] + b_2[V(2)] + \cdots + b_n[V(n)]$$

with integer coefficients b_i with multiplicative structure to be explained. The weight space decomposition of $V(m)$ gives

$$\mathrm{ch}_{V(m)} = t^m + t^{m-2} + t^{m-4} + \cdots + t^{-m} = \frac{t^{m+1} - t^{-m-1}}{t - t^{-1}}$$

Using the fact that ch is a ring isomorphism we get:

$$\mathrm{ch}_{V(n)\otimes V(m)} = \mathrm{ch}_{V(n)}\mathrm{ch}_{V(m)} = \frac{(t^{n+1}-t^{-n-1})(t^{m+1}-t^{-m-1})}{(t-t^{-1})^2}$$
$$= \frac{t^{m+n+2}-t^{m-n}+t^{-m-n-2}-t^{n-m}}{(t-t^{-1})^2}$$

This implies the following.

Theorem 22.3.3 (Clebsch-Gordon)**.** *If* $m \geq n$ *then*

$$V(n)\otimes V(m) \cong V(n+m)+V(n+m-2)+V(n+m-4)+\cdots+V(m-n)$$

Proof. The formal character of the right hand side is:

$$\frac{t^{m+n+1}-t^{-m-n-1}}{t-t^{-1}}+\frac{t^{m+n-1}-t^{-m-n+1}}{t-t^{-1}}+\cdots+\frac{t^{m-n+1}-t^{-m+n-1}}{t-t^{-1}}$$

which is equal to the formal character of $V(n)\otimes V(m)$ given above. □

22.4. **Weyl character formula.** Recall that $\delta \in \Lambda$ is the integral weight given by

$$\delta = \sum \lambda_i = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$$

Definition 22.4.1.

$$q = \sum_{\sigma\in W}\mathrm{sgn}(\sigma)e(\sigma\delta)\in\mathbb{Z}[\Lambda]$$

where $\mathrm{sgn}(\sigma) = (-1)^\ell$ where $\ell = \ell(\sigma)$ is the length of σ (the minimum number of reflections that gives σ. (This is also $\mathrm{sgn}(\sigma) = \det(\sigma)$ since each reflection has $\det = -1$.)

Example 22.4.2. For $L = \mathfrak{sl}(2,F)$, $\delta = \lambda_1 = \frac{1}{2}\alpha_1$ is equal to $t = t^1$ in the notation above. The nontrivial element of W is a reflection with sign -1. So, the element q is equal to

$$q = t - t^{-1}$$

Definition 22.4.3. For any integral weight μ let

$$\omega(\mu) = \sum_{\sigma\in W}\mathrm{sgn}(\sigma)e(\sigma\mu)\in\mathbb{Z}[\Lambda]$$

Example 22.4.4. For $L = \mathfrak{sl}(2,F)$, $e(m) = t^m$ and $\omega(m) = t^m - t^{-m}$. In particular, $q = \omega(\delta) = \omega(1) = t - t^{-1}$.

Theorem 22.4.5 (Weyl Character Formula)**.** *For any dominant weight* $\lambda \in \Lambda^+$*, the formal character of* $V(\lambda)$ *is given by*

$$\boxed{\mathrm{ch}_\lambda = \frac{\omega(\lambda+\delta)}{\omega(\delta)}}$$

Example 22.4.6. For $\mathfrak{sl}(2,F)$ this gives

$$\mathrm{ch}_{V(m)} = \frac{\omega(m+1)}{\omega(1)} = \frac{t^{m+1}-t^{-m-1}}{t-t^{-1}}$$