22.5. **Example in $\mathfrak{sl}(3,F)$.** We will examine what the Weyl character formula says in the case of $\mathfrak{sl}(n+1,F)$ (with rank $=\dim H = n$). First, we continue the "superman" example from before for $\mathfrak{sl}(3,F)$. In this case $W = S_3$ has six elements, three have positive sign and three have negative sign. This means that

$$\omega(\lambda+\delta) = \sum_{\sigma\in W=S_3} \operatorname{sgn}(\sigma) e(\sigma(\lambda+\delta))$$

has six terms, three positive and three negative as shown in red in Figure 22.5.1. The support of ch_λ for $\lambda = 3\lambda_1 + 2\lambda_2$ is shown in blue.

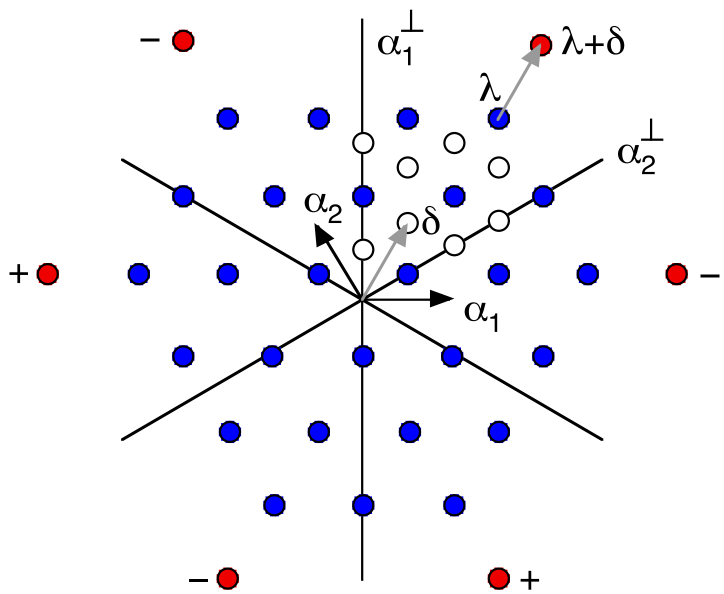


FIGURE 22.5.1. Support of ch_λ for $\lambda = 3\lambda_1 + 2\lambda_2$ in blue. The white dots are the other dominant weights including $\delta = \lambda_1 + \lambda_2$. They lie in the fundamental chamber between $\alpha_1^\perp$ and $\alpha_2^\perp$. Support of $\omega(\lambda+\delta)$ is in red.

Here we make the important observation that $\lambda+\delta$ is always strongly dominant. (So, the red spots are always distinct and do not cancel.)

The Weyl character formula says $ch_\lambda\,\omega(\delta) = \omega(\lambda+\delta)$ or

$$ch_\lambda \sum_{\sigma\in W} \operatorname{sgn}(\sigma) e(\sigma\delta) = \sum_{\sigma\in W} \operatorname{sgn}(\sigma) e(\sigma(\lambda+\delta))$$

Comparing the coefficient of $e(\mu)$ on both sides, we get:

$$\sum_{\sigma\in W} \operatorname{sgn}(\sigma) m_\lambda(\mu - \sigma\delta) = \begin{cases} \operatorname{sgn}(\sigma) & \text{if } \mu = \sigma(\lambda+\delta) \\ 0 & \text{otherwise} \end{cases}$$

For example, around the point marked μ ($= \lambda - \delta$) in Figure 22.5.2, the values of $\mathrm{sgn}(\sigma)m_\lambda(\mu - \sigma\delta)$ are

$$+3 - 2 + 1 - 1 + 1 - 2 = 0$$

Using the fact that the support of ch_λ is in the convex hull of the points $\sigma\lambda, \sigma \in W$ (the corners of the blue spot region) we can determine the value of $m_\lambda(\mu)$ at all points: starting with the value of $m_\lambda(\mu) = 1$ at the six corners (including $\mu = \lambda$) and working inward, we see that the value of $m_\lambda(\mu)$ is as given in Figure 22.5.2.

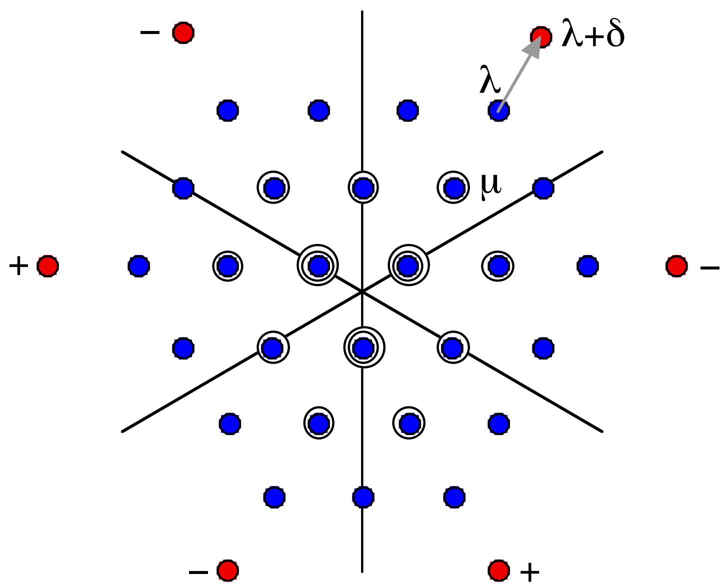


FIGURE 22.5.2. $m_\lambda(\mu)$ as given by the Weyl character formula.

22.6. **Character formula for $\mathfrak{sl}(n+1, F)$.** The simple roots of $L = \mathfrak{sl}(n+1, F)$ are $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $i = 1, \cdots, n$, where $\epsilon_i : H \to F$ is projection to the ith entry. The corresponding basis elements in H are $h_i = e_i - e_{i+1}$ where e_i is the diagonal matrix with 1 in the ith position and 0 elsewhere. Thus $\lambda = \sum_{i=1}^n a_i\epsilon_i$ is dominant iff $a_1 \geq a_2 \geq \cdots a_n \geq 0$. The fundamental dominant weights are:

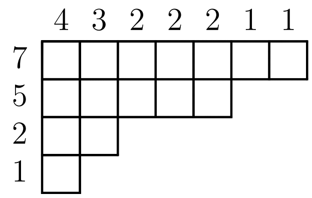
$\lambda_1 = \epsilon_1$

$\lambda_2 = \epsilon_1 + \epsilon_2$

$\lambda_i = \epsilon_1 + \cdots + \epsilon_i$

For example, $\lambda = 2\lambda_1 + 3\lambda_2 + \lambda_3 + \lambda_4$ is equal to $7\epsilon_1 + 5\epsilon_2 + 2\epsilon_3 + \epsilon_4$. This is illustrated by the Young diagram below. Also, $\delta = \sum \lambda_i = \sum(n+1-i)\epsilon_i$. Thus, if $n = 5$, then

$$\lambda + \delta = (7, 5, 2, 1, 0) + (5, 4, 3, 2, 1) = 12\epsilon_1 + 9\epsilon_2 + 5\epsilon_3 + 3\epsilon_4 + \epsilon_5$$



The Weyl group $W = S_{n+1}$ permutes the elements ϵ_i, $i = 1, \cdots, n+1$. The weights μ which occur in ch_λ for $\lambda = \sum a_i \epsilon_i$ correspond to points $(b_1, \cdots, b_{n+1})$ in the affine n plane given by $\sum b_i = \sum a_i$ (= 15 in this case). The correspondence is given by

$$\mu(b) = \sum_{i=1}^{n} (b_i - b_{i+1})\lambda_i$$

This correspondence is additive in the sense that $\mu(b + b') = \mu(b) + \mu(b')$.

Theorem 22.6.1. *The coefficient $m_\lambda(\mu)$ of $e(\mu)$ in ch_λ for $\mu = \mu(b)$ as above is equal to the coefficient of $\prod x_i^{b_i}$ in the Schur polynomial $s_\lambda(x_1, \cdots, x_{n+1})$.*

The proof of this is simply that the definition of the Schur polynomial gives the Weyl character formula under the multiplicative correspondence

$$e(\mu(b)) \leftrightarrow \prod x_i^{b_i}$$

22.7. **Schur polynomials.**

Definition 22.7.1. If $\lambda = (a_1, \cdots, a_n)$ where $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ then

$$s_\lambda(x_1, \cdots, x_{n+1}) := \frac{W(x_1^{a_1+n} x_2^{a_2+n-1} \cdots x_n^{a_n+1})}{W(x_1^n x_2^{n-1} \cdots x_n)}$$

where

$$W(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) = \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) x_{\sigma(1)}^{a_1} \cdots x_{\sigma(n)}^{a_n}$$

Remark 22.7.2. The components of λ are usually called $\lambda_1 \geq \lambda_2 \geq \cdots$. However, we have a clash of notation since λ_i is the ith fundamental dominant weight. So we use a_i.

The numerator and denominator of s_λ are homogeneous *alternating polynomials* in the sense that

$$W(x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \cdots x_{\sigma(n)}^{a_n}) = \operatorname{sgn}(\sigma) W(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n})$$

Furthermore, the denominator is:

$$\Delta := W(x_1^n x_2^{n-1} \cdots x_n) = \prod_{1 \leq i < j \leq n+1} (x_i - x_j)$$

which has the property that it divides every alternating polynomial since $x_i - x_j$ clearly divides every alternating polynomial. Therefore, the ratio is a homogeneous *symmetric polynomial* in the sense that

$$s_\lambda(x_{\sigma(1)}, \cdots, x_{\sigma(n+1)}) = s_\lambda(x_1, \cdots, x_{n+1})$$

for all $\sigma \in S_{n+1}$.

Example 22.7.3. In the first example we have $n = 2, \lambda = 3\lambda_1 + 2\lambda_2 = 5\epsilon_1 + 2\epsilon_2$ (and $\lambda + \delta = 7\epsilon_1 + 3\epsilon_2$). So,

$$s_{(5,2)}(x_1, x_2, x_3) = \frac{W(x_1^7 x_2^3)}{W(x_1^2 x_2)} = \frac{x_1^7x_2^3 - x_1^3x_2^7 + x_2^7x_3^3 - x_2^3x_3^7 + x_1^3x_3^7 - x_1^7x_3^3}{x_1^2x_2 - x_1x_2^2 + x_2^2x_3 - x_2x_3^2 + x_1x_3^2 - x_1^2x_3}$$

This is a homogeneous polynomial of degree 7 in which every monomial corresponds to a weight $\mu(b) = \sum(b_i - b_{i+1})\lambda_i$ of $V(\lambda)$ with $b_1 + b_2 + b_3 = 7$ as illustrated in Figure 22.7 below.

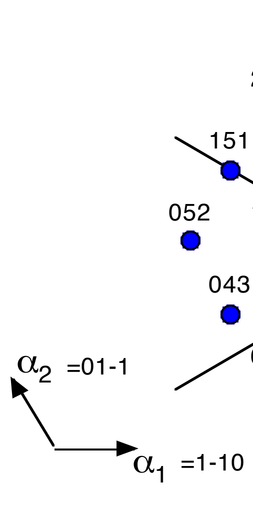


Figure 22.7.1. The points (b_1, b_2, b_3) with nonnegative integer b_i are in the plane given by $\sum b_i = 7$. The Weyl group $W = S_3$ permutes the b_i.

Figure (22.7) illustrates the equation: $s_\lambda(x_1, x_2, x_3) =$

$$x_1^2x_2^5 \quad + x_1^3x_2^4 \quad + x_1^4x_2^3 \quad + x_1^5x_2^2$$

$$+x_1x_2^5x_3 + 2x_1^2x_2^4x_3 + 2x_1^3x_2^3x_3 + 2x_1^4x_2^2x_3 + x_1^5x_2x_3$$

$$+x_2^5x_3^2 + 2x_1x_2^4x_3^2 + 3x_1^2x_2^3x_3^2 + 3x_1^3x_2^2x_3^2 + 2x_1^4x_2x_3^2 + x_1^5x_3^2$$

$$+x_2^4x_3^3 + 2x_1x_2^3x_3^3 + 3x_1^2x_2^2x_3^3 + 2x_1^3x_2x_3^3 + x_1^4x_3^3$$

$$+x_2^3x_3^4 + 2x_1x_2^2x_3^4 + 2x_1^2x_2x_3^4 + x_1^3x_3^4$$

$$+x_2^2x_3^5 \ + x_1x_2x_3^5 \ + x_1^2x_3^5$$

22.8. **Dimension formula.**

Theorem 22.8.1. *For any dominant weight λ, the dimension of the irreducible module $V(\lambda)$ is given by*

$$\dim V(\lambda) = \prod_{\sigma\in\Phi_+} \frac{\langle\lambda+\delta,\alpha\rangle}{\langle\delta,\alpha\rangle} = \prod_{\sigma\in\Phi_+} \frac{(\lambda+\delta,\alpha)}{(\delta,\alpha)}$$

In the case $L = \mathfrak{sl}(n+1, F)$ we get:

Corollary 22.8.2. *If $\lambda = (a_1, \cdots, a_{n+1})$ where $a_1 \geq \cdots \geq a_n \geq a_{n+1} = 0$ then*

$$s_\lambda(1,1,\cdots,1) = \prod_{1\leq i<j\leq n+1} \frac{a_i - a_j + j - i}{j-i}$$

Proof. This follows from the fact that the positive roots of type A_n are $\alpha_{ij} = \epsilon_i - \epsilon_j$ and $\langle\lambda, \alpha_{ij}\rangle = a_i - a_j$. □

Example 22.8.3. For $n = 2$ and $\lambda = (5, 2, 0)$ we have:

$$s_{(5,2)}(1,1,1) = \frac{1}{2}(a_1 - a_3 + 2)(a_1 - a_2 + 1)(a_2 - a_3 + 1)$$
$$\frac{1}{2}(5-0+2)(5-2+1)(2-0+1) = \frac{7\cdot 4\cdot 3}{2} = 42$$

which is equal to the number of blue spots with multiplicity calculated above.

Proof of Theorem 22.8.1. Let $v : \mathbb{Z}[\Lambda] \to \mathbb{Z}$ be the *augmentation map* given by $v(e(\mu)) = 1$ for all $\mu \in \Lambda$. Then

$$\dim V(\lambda) = v(ch_\lambda)$$

However, the augmentation of

$$\omega(\delta) = \sum_{\sigma\in W} \mathrm{sgn}(\sigma) e(\sigma\delta) = \prod_{\alpha\in\Phi_+} \left(e\left(\frac{\alpha}{2}\right) - e\left(-\frac{\alpha}{2}\right)\right)$$

is zero. So, we get $\dim V(\lambda) = \frac{0}{0}$.

The equation $e(\lambda+\mu) = e(\lambda)e(\mu)$ implies that, for all $\alpha \in \Phi_+$,

$$\partial_\alpha e(\lambda) := (\lambda, \alpha)e(\lambda)$$

is a derivation. Take the equation

$$ch_\lambda\,\omega(\delta) = \omega(\lambda+\delta)$$

and apply the product $\partial = \prod_{\alpha\in\Phi_+} \partial_\alpha$ of all of these derivations. Then take the augmentation v.

Claim: For any nonempty subset $S \subset \Phi_+$, $v\prod_{\alpha\in S} \partial_a \omega(\delta) = 0$.

This follows from the fact that $\omega(\delta)$ is a product of $|\Phi_+|$ factors each of which evaluates to 0. So, we must differentiate each factor at least once. This gives:

$$\partial\omega(\delta) = \sum_{\sigma\in W} \mathrm{sgn}(\sigma) \prod_{\alpha\in\Phi_+} \langle\sigma\delta, \alpha\rangle\, e(\sigma\delta)$$

$$= \sum_{\sigma \in W} \operatorname{sgn}(\sigma) \prod_{\alpha \in \Phi_+} \left\langle \delta, \sigma^{-1}\alpha \right\rangle e(\sigma\delta)$$

However, the collection of roots $\{\sigma^{-1}\alpha\}$ is, up to sign, the set of positive roots with exactly $\ell(\alpha)$ negative roots. Since $\operatorname{sgn}(\sigma) = (-1)^{\ell(\sigma)}$, for each $\sigma \in W$ we have:

$$\operatorname{sgn}(\sigma) \prod_{\alpha \in \Phi_+} \left\langle \delta, \sigma^{-1}\alpha \right\rangle = \prod_{\alpha \in \Phi_+} \langle \delta, \alpha \rangle$$

Therefore,

$$v(\omega(\delta)) = |\Phi_+| \prod_{\alpha \in \Phi_+} \langle \delta, \alpha \rangle$$

Similarly,

$$v(\omega(\lambda + \delta) = |\Phi_+| \prod_{\alpha \in \Phi_+} \langle \lambda + \delta, \alpha \rangle$$

This gives

$$\dim V(\lambda) |\Phi_+| \prod_{\alpha \in \Phi_+} \langle \delta, \alpha \rangle = |\Phi_+| \prod_{\alpha \in \Phi_+} \langle \lambda + \delta, \alpha \rangle$$

and the theorem follows. □