23. Schur functors

In this lecture we will examine the Schur polynomials and Young tableaux to obtain irreducible representations of $\mathfrak{sl}(n+1,\mathbb{C})$. The construction is based on Fulton's book "Young Tableaux" which is highly recommended for beginning students because it is a beautifully written book with minimal prerequisites. This will be only in the case of $L = \mathfrak{sl}(n+1, F)$. Other cases and the proof of the Weyl character formula will be next.

First, we construct the irreducible representations $V(\lambda_i)$ corresponding to the fundamental dominant weights $\lambda_i = \epsilon_1 + \cdots + \epsilon_i$. Then we use Young tableaux to find the modules $V(\lambda)$ inside tensor products of these fundamental representations.

23.1. **Fundamental representations.**

Theorem 23.1.1. *The fundamental representation $V(\lambda_j)$ is equal to the ith exterior power of $V = F^{n+1}$:*

$$V(\lambda_j) = \wedge^j V$$

$L = \mathfrak{sl}(n+1, F)$ acts by the isomorphism $\mathfrak{sl}(n+1,F) \cong \mathfrak{sl}(V)$.

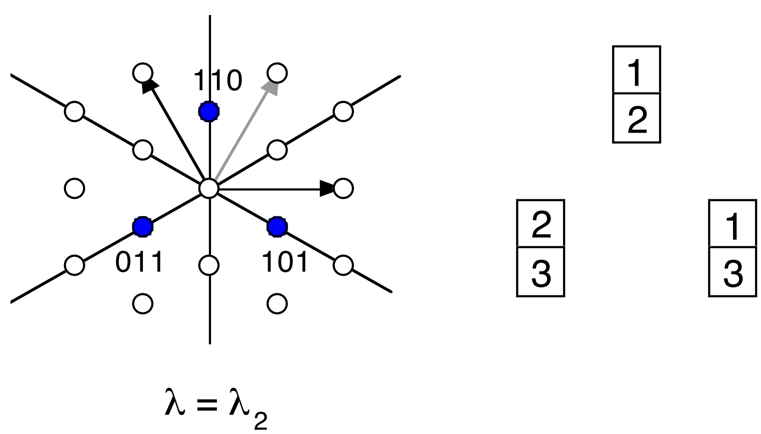
Definition 23.1.2. Recall that the *jth exterior power* of a vector space V is the quotient of the jth tensor power by all elements of the form $w_1 \otimes \cdots \otimes w_j$ where the w_i are not distinct. The image of $w_1 \otimes \cdots \otimes w_j$ in $\wedge^j V$ is denoted $w_1 \wedge \cdots \wedge w_j$. When the characteristic of F is not equal to 2, this is equivalent to saying that the wedge is skew symmetric:

$$w_{\sigma(1)} \wedge \cdots \wedge w_{\sigma(j)} = \operatorname{sgn}(\sigma) w_1 \wedge \cdots \wedge w_j$$

If $v_1, \cdots, v_{n+1}$ is a basis for V then

$$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_j}$$

for $1 \le i_1 < i_2 < \cdots < i_j \le n+1$ is a basis for $\wedge^j V$.



Example 23.1.3. Take $n = 2$ and $\lambda = \lambda_2 = \epsilon_1 + \epsilon_2$. The Schur polynomial is

$$s_{11} = x_1x_2 + x_2x_3 + x_1x_3$$

The Young diagram is filled in with numbers $1, \cdots, n+1$ so that the numbers are increasing as we go down and nondecreasing as we go across. Let $V = F^{n+1} = F^3$ in this case with basis v_1, v_2, v_3. Then the corresponding representation is $\wedge^2 V$ with basis

$$v_1 \wedge v_2, \quad v_2 \wedge v_3, \quad v_1 \wedge v_3$$

The action of the Cartan subalgebra H is given on basis elements by

$$h(v_i \wedge v_j) = h(v_i) \wedge v_j + v_i \wedge h(v_j) = \epsilon_i(h)v_i \wedge v_j + v_i \wedge \epsilon_j(h)v_j$$

Therefore, $v_i \wedge v_j$ lies in the $\epsilon_i + \epsilon_j$ weight space. Since $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ we get:

vector	$v_1 \wedge v_2$	$v_2 \wedge v_3$	$v_1 \wedge v_3$
	$\epsilon_1 + \epsilon_2$	$\epsilon_2 + \epsilon_3$	$\epsilon_1 + \epsilon_3$
weight	λ_2	$-\lambda_1$	$\lambda_1 - \lambda_2$
	maximal		

Proof of Theorem. In the general case, the maximal vector in $\wedge^j V$ is $v_1 \wedge v_2 \wedge \cdots \wedge v_j$ with highest weight λ_j. □

One of the key points is that the wedge $V \mapsto \wedge^j V$ is a *functor* (i.e., natural). In particular, any endomorphism $g : V \to V$ induces an endomorphism $\wedge^j g : \wedge^j V \to \wedge^j V$ making $\wedge^j V$ into a module over $\mathfrak{sl}(V) \cong \mathfrak{sl}(n+1, F)$.

23.2. **Symmetric powers.** As I pointed out before, the irreducible module $V(\lambda)$ for $\lambda = \sum c_i\lambda_i$ is a direct summand of a tensor product of $V(\lambda_i)$, taking c_i copies of $V(\lambda_i)$. For example, take $n = 2$ and $\lambda = \lambda_1 + \lambda_2 = 2\epsilon_1 + \epsilon_2$. But, the tensor product of $V(\lambda_2) = \wedge^2 V$ and $V(\lambda_1) = \wedge^1 V = V$ is $3 \times 3 = 9$ dimensional, whereas, $V(\lambda_1 + \lambda_2)$ is 8 dimensional (see worksheet). The Schur-Weyl construction tells us which 1-dimensional subspace to mod out.

Example 23.2.1. Take the example $\lambda = 2\lambda_1$. This Young diagram

and Schur polynomial

$$s_{20} = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3$$

This is the sum of all monomials of degree 2 in the variables x_1, x_2, x_3. Therefore, it corresponds to the second symmetric power of V:

$$V(2\lambda_1) = \mathcal{S}^2(V)$$

A basis for $\mathcal{S}^2(V)$ is given by $v_i \widetilde{\otimes} v_j$ where $1 \leq i \leq j \leq n+1$. These basis elements are represented by filling in the boxes:

i	j

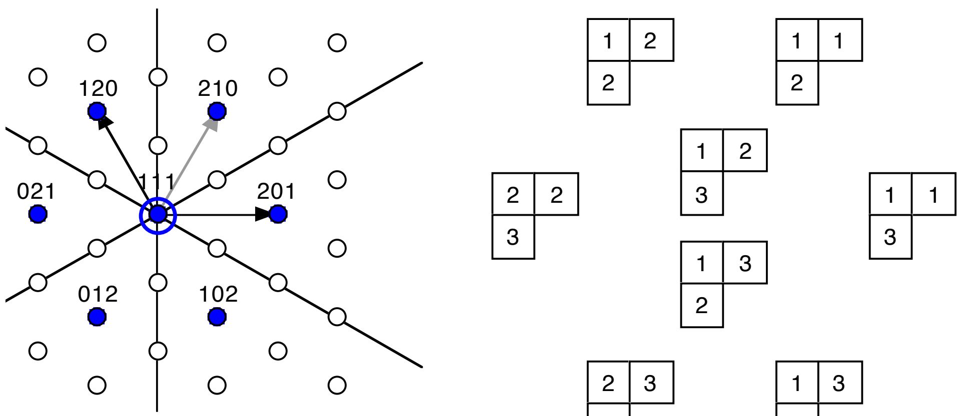
More generally, for any n and k, we have

$$V(k\lambda_1) = \mathcal{S}^k(V)$$

23.3. **Schur functors.** Let $\lambda = \sum \lambda_{b_j}$, where $b_1 \geq b_2 \geq \cdots$, be given by a Young diagram D (whose jth column has b_j boxes). Then the Schur functor $\mathcal{S}_\lambda(V)$ is given as a quotient of

$$\wedge^{b_1} V \otimes \wedge^{b_2} V \otimes \wedge^{b_3} V \otimes \cdots$$

by *exchange relation* which I will explain in class using the two examples on the worksheet.



$\lambda = \lambda_1 + \lambda_2$

Example 23.3.1. Take the example $\lambda = \lambda_1 + \lambda_2 = 2\epsilon_1 + \epsilon_2$. The irreducible module $V(\lambda)$, which is 8 dimensional by the calculation in the diagrams above, is a direct summand of $V(\lambda_2) \otimes V(\lambda_1) = \wedge^2 V \otimes \wedge^1 V = \wedge^2 V \otimes V$ which is $3 \times 3 = 9$ dimensional. What is the missing basis vector?

The missing vector is $(v_2 \wedge v_3) \otimes v_1$ which corresponds to the Young tableaux

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1} \end{array}$$

This is not admissible since $2 > 1$ but it is a linear combination of admissible Young tableaux by the exchange relation which says that the contents of any box can be exchanged with the contents of all boxed in any other column:

$$\begin{array}{|c|c|}\hline A & D \\ \hline B \\ \cline{1-1} C \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline D & A \\ \hline B \\ \cline{1-1} C \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline A & B \\ \hline D \\ \cline{1-1} C \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline A & C \\ \hline B \\ \cline{1-1} D \\ \cline{1-1} \end{array}$$

In this case we have:

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1} \end{array}$$

The second term is equal to

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1} \end{array} = - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

since the first column represents $v_2 \wedge v_1 = -v_1 \wedge v_2$. Thus:

$$(v_2 \wedge v_3)\tilde{\otimes} v_1 = (v_1 \wedge v_3)\tilde{\otimes} v_2 - (v_1 \wedge v_2)\tilde{\otimes} v_3$$

where $\tilde{\otimes}$ indicates tensor product symmetrised by the exchange relations.

More generally, the exchange relation says that any set of squares in any column can be exchanged with squares in one other column as long as one takes the sum of all ways to do that and, also, the second column is to the left of the first column. For example, if two columns have the same number of squares then they can be switched. Another example is:

$$\begin{array}{|c|c|} \hline A & D \\ \hline B & E \\ \hline C & \\ \cline{1-1} \end{array} = \begin{array}{|c|c|} \hline D & A \\ \hline E & B \\ \hline C & \\ \cline{1-1} \end{array} + \begin{array}{|c|c|} \hline A & B \\ \hline D & C \\ \hline E & \\ \cline{1-1} \end{array} + \begin{array}{|c|c|} \hline D & A \\ \hline B & C \\ \hline E & \\ \cline{1-1} \end{array}$$

Theorem 23.3.2. *If λ is the sum of fundamental dominant weights λ_{j_i} then $V(\lambda)$ is the quotient of $\bigotimes V(\lambda_{j_i})$ by the exchange relations.*

This quotient is the *Schur functor* $\mathcal{S}_\lambda(V)$.

Proof. The quotient $\mathcal{S}_\lambda(V)$ is a representation of $L = \mathfrak{sl}(V)$. By a combinatorial argument, we can see that it has a basis given by the admissible Young tableaux. The Schur polynomial s_λ is the sum of the corresponding monomials. So, the character of $\mathcal{S}_\lambda(V)$ is equal to ch_λ as given by the Weyl character formula. But we know that representations are uniquely determined by their formal characters. So, we conclude that $\mathcal{S}_\lambda(V) = V(\lambda)$. $\square$