23. Schur functors

In this lecture we will examine the Schur polynomials and Young tableaux to obtain irreducible representations of $\mathfrak{sl}(n+1,\mathbb{C})$. The construction is based on Fulton's book "Young Tableaux" which is highly recommended for beginning students because it is a beautifully written book with minimal prerequisites. This will be only in the case of $L = \mathfrak{sl}(n+1,F)$. Other cases and the proof of the Weyl character formula will be next.

First, we construct the irreducible representations $V(\lambda_i)$ corresponding to the fundamental dominant weights $\lambda_i = \epsilon_1 + \cdots + \epsilon_i$. Then we use Young tableaux to find the modules $V(\lambda)$ inside tensor products of these fundamental representations.

23.1. **Fundamental representations.**

Theorem 23.1.1. *The fundamental representation $V(\lambda_j)$ is equal to the ith exterior power of $V = F^{n+1}$:*

$$V(\lambda_j) = \wedge^j V$$

$L = \mathfrak{sl}(n+1,F)$ acts by the isomorphism $\mathfrak{sl}(n+1,F) \cong \mathfrak{sl}(V)$.

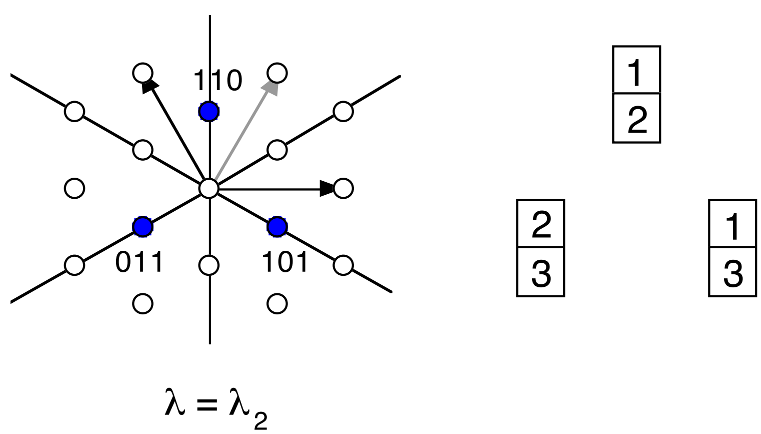
Definition 23.1.2. Recall that the *jth exterior power* of a vector space V is the quotient of the jth tensor power by all elements of the form $w_1 \otimes \cdots \otimes w_j$ where the w_i are not distinct. The image of $w_1 \otimes \cdots \otimes w_j$ in $\wedge^j V$ is denoted $w_1 \wedge \cdots \wedge w_j$. When the characteristic of F is not equal to 2, this is equivalent to saying that the wedge is skew symmetric:

$$w_{\sigma(1)} \wedge \cdots \wedge w_{\sigma(j)} = \operatorname{sgn}(\sigma) w_1 \wedge \cdots \wedge w_j$$

If $v_1, \cdots, v_{n+1}$ is a basis for V then

$$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_j}$$

for $1 \le i_1 < i_2 < \cdots < i_j \le n+1$ is a basis for $\wedge^j V$.



Example 23.1.3. Take $n = 2$ and $\lambda = \lambda_2 = \epsilon_1 + \epsilon_2$. The Schur polynomial is

$$s_{11} = x_1x_2 + x_2x_3 + x_1x_3$$

The Young diagram is filled in with numbers $1, \cdots, n{+}1$ so that the numbers are increasing as we go down and nondecreasing as we go across. Let $V = F^{n+1} = F^3$ in this case with basis v_1, v_2, v_3. Then the corresponding representation is $\wedge^2 V$ with basis

$$v_1 \wedge v_2, \quad v_2 \wedge v_3, \quad v_1 \wedge v_3$$

The action of the Cartan subalgebra H is given on basis elements by

$$h(v_i \wedge v_j) = h(v_i) \wedge v_j + v_i \wedge h(v_j) = \epsilon_i(h)v_i \wedge v_j + v_i \wedge \epsilon_j(h)v_j$$

Therefore, $v_i \wedge v_j$ lies in the $\epsilon_i + \epsilon_j$ weight space. Since $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ we get:

vector	$v_1 \wedge v_2$	$v_2 \wedge v_3$	$v_1 \wedge v_3$
	$\epsilon_1 + \epsilon_2$	$\epsilon_2 + \epsilon_3$	$\epsilon_1 + \epsilon_3$
weight	λ_2	$-\lambda_1$	$\lambda_1 - \lambda_2$
	maximal		

Proof of Theorem. In the general case, the maximal vector in $\wedge^j V$ is $v_1 \wedge v_2 \wedge \cdots \wedge v_j$ with highest weight λ_j. □

One of the key points is that the wedge $V \mapsto \wedge^j V$ is a *functor* (i.e., natural). In particular, any endomorphism $g : V \to V$ induces an endomorphism $\wedge^j g : \wedge^j V \to \wedge^j V$ making $\wedge^j V$ into a module over $\mathfrak{sl}(V) \cong \mathfrak{sl}(n+1, F)$.

23.2. **Symmetric powers.** As I pointed out before, the irreducible module $V(\lambda)$ for $\lambda = \sum c_i\lambda_i$ is a direct summand of a tensor product of $V(\lambda_i)$, taking c_i copies of $V(\lambda_i)$. For example, take $n = 2$ and $\lambda = \lambda_1 + \lambda_2 = 2\epsilon_1 + \epsilon_2$. But, the tensor product of $V(\lambda_2) = \wedge^2 V$ and $V(\lambda_1) = \wedge^1 V = V$ is $3 \times 3 = 9$ dimensional, whereas, $V(\lambda_1 + \lambda_2)$ is 8 dimensional (see worksheet). The Schur-Weyl construction tells us which 1-dimensional subspace to mod out.

Example 23.2.1. Take the example $\lambda = 2\lambda_1$. This Young diagram

and Schur polynomial

$$s_{20} = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3$$

This is the sum of all monomials of degree 2 in the variables x_1, x_2, x_3. Therefore, it corresponds to the second symmetric power of V:

$$V(2\lambda_1) = \mathcal{S}^2(V)$$

A basis for $\mathcal{S}^2(V)$ is given by $v_i \otimes v_j$ where $1 \leq i \leq j \leq n + 1$. These basis elements are represented by filling in the boxes:

i	j

More generally, for any n and k, we have

$$V(k\lambda_1) = \mathcal{S}^k(V)$$

23.3. **Schur functors.** Let $\lambda = \sum \lambda_{b_j}$, where $b_1 \geq b_2 \geq \cdots$, be given by a Young diagram D (whose jth column has b_j boxes). Then the Schur functor $\mathcal{S}_\lambda(V)$ is given as a quotient of

$$\wedge^{b_1} V \otimes \wedge^{b_2} V \otimes \wedge^{b_3} V \otimes \cdots$$

by *exchange relation* which I will explain in class using the two examples on the worksheet.