24. Proof of Weyl character formula

(1) Casimir operator
(2) Review α-root strings.
(3) Formula for $m_V(\mu) = \dim V_\mu$
(4) Freudenthal multiplicity formula
(5) Weyl character formula
(6) Kostant's formula

This proof is from Fulton and Harris, Lecture 25, following the notation of Humphreys Section 22.

24.1. **Casimir operator.**

(from subsection 6.3). If V is a representation of a semisimple Lie algebra L we have a homomorphism $\varphi : L \to \mathfrak{gl}(V)$. The *Casimir operator* $c \in \mathrm{End}_L(V)$ of φ is given by

$$c_V = \sum \varphi(x_i)\varphi(x_i^*)$$

where $\{x_i\}$ is a basis for L and $\{x_i^*\}$ is the dual basis with respect to the Killing form κ. We proved the following:

(1) c_V is independent of the choice of basis $\{x_i\}$.
(2) $c_V : V \to V$ is a homomorphism of L-modules.
(3) If V is irreducible, then, by Schur's Lemma, c_V is multiplication by a scalar (which we also call c).

Lemma 24.1.1. *If V is irreducible, the trace of $c_V = \sum \varphi(x_i)\varphi(x_i^*)|V_\mu$ is $c\, m_V(\mu) = c \dim V_\mu$.*

24.1.1. *choice of basis.* Take a root space decomposition $L = H \oplus \bigoplus L_\alpha$ and take the basis $h_i, i = 1, \cdots, n$ for H. For every root $\alpha \in \Phi$ choose an element $x_\alpha \in L_\alpha$. This gives a basis for L.

Let $\{h_i^*\}$ be the basis of H dual to $\{h_i\}$.

Lemma 24.1.2.

$$\sum_{i=1}^n \mathrm{Tr}(\varphi(h_i)\varphi(h_i^*)|V_\mu) = \sum_{i=1}^n \mu(h_i)\mu(h_i^*)m_V(\mu) = (\mu,\mu)m_V(\mu)$$

Proof. Suppose $t_\mu = \sum a_i h_i$. By definition of t_μ we have

$$\mu(h_j) = \kappa(t_\mu, h_j) = \sum_i a_i \kappa(h_i, h_j)$$

$$\mu(h_j^*) = \sum_i a_i \kappa(h_i, h_j^*) = a_j$$

So,

$$\sum_j \mu(h_j)\mu(h_j^*) = \sum_{i,j} a_i a_j \kappa(h_i, h_j) = \kappa(t_\mu, t_m u) = (\mu,\mu)$$

□

Lemma 24.1.3. *Let* z_α *be dual to* x_α*. (So,* $\kappa(x_\alpha, z_\alpha) = 1$*.) Then*

$$z_\alpha = \frac{(\alpha,\alpha)}{2} y_\alpha \qquad \text{and} \qquad [x_\alpha, z_\alpha] = t_\alpha$$

Proof. Since $z_\alpha \in L_{-\alpha}$ which is 1-dimensional, and

$$[x_\alpha, y_\alpha] = h_\alpha = \frac{2}{(\alpha,\alpha)} t_\alpha$$

the two equations are equivalent. Associativity of κ forces the second equation to be true:

$$\kappa(h_\alpha, [x_\alpha, z_\alpha]) = \kappa([h_\alpha, x_\alpha], z_\alpha) = \kappa(2x_\alpha, z_\alpha) = 2$$

$$\kappa(h_\alpha, t_\alpha) = \alpha(h_\alpha) = 2.$$

Since $[x_\alpha, z_\alpha]$ is a multiple of t_α, they must be equal. □

Lemma 24.1.4.

$$\mathrm{Tr}(\varphi(x_\alpha)\varphi(z_\alpha)|V_\mu) = \sum_{i=0}^{\infty} (\mu + i\alpha, \alpha) m_V(\mu + i\alpha)$$

Multiplying both sides by $\frac{2}{(\alpha,\alpha)}$ we see that this is equivalent to:

$$\mathrm{Tr}(\varphi(x_\alpha)\varphi(y_\alpha)|V_\mu) = \sum_{i=0}^{\infty} \langle \mu + i\alpha, \alpha \rangle\, m_V(\mu + i\alpha) \tag{24.1}$$

To prove this we need to look at the α root string through μ.

24.2. **α-root strings.** Recall (subsection 8.4.2) that the α string through μ is

$$V_{\mu+q\alpha} \oplus V_{\mu+(q-1)\alpha} \oplus \cdots \oplus V_{\mu+\alpha} \oplus V_\mu \oplus V_{\mu-\alpha} \oplus \cdots \oplus V_{\mu-r\alpha}$$

where $\mu(h_\alpha) = \langle \mu, \alpha \rangle = r - q$ and

$$q + r = m = (\mu + q\alpha)(h_\alpha) = \langle \mu + q\alpha, \alpha \rangle = \langle \mu, \alpha \rangle + 2q$$

Consider the α string as a module over $S_\alpha \cong \mathfrak{sl}(2, F)$. Then $m_V(\mu + q\alpha) = \dim V_{\mu+q\alpha}$ is the number of copies of the indecomposable S_α module $V(m)$ in V where $V(m)$ is the S_α module which starts at $V_{\mu+q\alpha}$ and goes to $V_{\mu-r\alpha}$.

By Lemma 7.0.4, the action of $x_\alpha y_\alpha$ on the weight space $V(m)_\mu$ is

$$(q+1)(m-q) = (q+1)(\langle \mu, \alpha \rangle + q)$$

Since this occurs $m_V(\mu + q\alpha)$ times, the contribution of $V(m)$ to the trace of $\varphi(x_\alpha)\varphi(y_\alpha)$ is

$$m_V(\mu + q\alpha)(q+1)(\langle \mu, \alpha \rangle + q)$$

Consider the components of the α string which go from $V_{\mu+i\alpha}$ to $V_{\mu-j\alpha}$. The number of such components is $m_V(\mu + i\alpha) - m_V(\mu + (i+1)\alpha)$. Here $m = \langle \mu, \alpha \rangle + 2i$. So, the action of $x_\alpha y_\alpha$ on the weight space $V(m)_\mu$ is

$$(i+1)(m-i) = (i+1)(\langle \mu, \alpha \rangle + i)$$

Claim:

$$\mathrm{Tr}(\varphi(x_\alpha)\varphi(y_\alpha)|V_\mu) = \sum_{i=0}^{\infty}(i+1)(\langle\mu,\alpha\rangle + i)[m_V(\mu+i\alpha) - m_V(\mu+(i+1)\alpha)]$$

Furthermore, this expression simplifies to give Equation (24.1) since the coefficient of $m_V(\mu+i\alpha)$ is

$$(i+1)(\langle\mu,\alpha\rangle + i) - i(\langle\mu,\alpha\rangle + i - 1) = (\langle\mu,\alpha\rangle + i) + i = \langle\mu+i\alpha,\alpha\rangle$$

To prove the claim we note two things.

(1) The terms with $i > q$ are zero since $m_V(\mu+i\alpha) = 0$ in those cases.
(2) The terms corresponding to α strings which do not contain μ add up to zero.

To prove the second point, we look at the case where the α string goes from $V_{\mu+i\alpha}$ to $V_{\mu+(j+1)\alpha}$ where $j \geq 0$. The number of copies of this α string in V is

$$[m_V(\mu+i\alpha) - m_V(\mu+(i+1)\alpha)]$$
$$= -[m_V(\mu+j\alpha) - m_V(\mu+(j+1)\alpha)]$$

Also, $\langle\mu,\alpha\rangle = -i-j-1$. This implies that the $i=i$ and $i=j$ terms in the summation cancel:

$$(i+1)(-i-j-1+i)[m_V(\mu+i\alpha) - m_V(\mu+(i+1)\alpha)]$$
$$+(j+1)(-i-j-1+j)[m_V(\mu+j\alpha) - m_V(\mu+(j+1)\alpha)] = 0$$

Point (2) also applies to the case where μ is not in the support of V (i.e., when $V_\mu = 0$). In that case, all terms in the sum will cancel. So, the formula in the Lemma applies to all μ.

Lemma 24.2.1. *For all weights μ we have:*

$$\sum_{i=-\infty}^{\infty}(\mu+i\alpha,\alpha)m_V(\mu+i\alpha) = 0$$

Proof. The sum does not change if we replace μ by $\mu - N\alpha$. Making N large enough, the sum will be zero for $i < 0$ and Point (2) applies. □

24.3. **First formula for $m_V(\mu)$.** Putting together the lemmas, we get the following for all weights μ.

$$c\, m_V(\mu) = \mathrm{Tr}(c_V|V_\mu) = (\mu,\mu)m_V(\mu) + \sum_{\alpha\in\Phi}\sum_{i=0}^{\infty}(\mu+i\alpha,\alpha)m_V(\mu+i\alpha)$$

The terms in the sum with $i = 0$ are $(\mu,\alpha)m_V(\mu)$ which changes sign when α becomes $-\alpha$. So, these terms cancel and we can start the summation with $i = 1$. By Lemma 24.2.1, the $-\alpha$ term in the sum is now

$$\sum_{i=1}^{\infty}(\mu-i\alpha,-\alpha)m_V(\mu-i\alpha) = \sum_{i=0}^{\infty}(\mu+i\alpha,\alpha)m_V(\mu+i\alpha)$$

This gives

$$c\,m_V(\mu) = (\mu,\mu)m_V(\mu) + \sum_{\alpha\in\Phi_+}(\mu,\alpha)m_V(\mu) + 2\sum_{\alpha\in\Phi_+}\sum_{i=1}^{\infty}(\mu+i\alpha,\alpha)m_V(\mu+i\alpha)$$

Using $\delta = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$, and collecting the $m_V(\mu)$ term on the left, we get:

$$(c-(\mu,\mu)-2(\mu,\delta))m_V(\mu) = 2\sum_{\alpha\in\Phi_+}\sum_{i=1}^{\infty}(\mu+i\alpha,\alpha)m_V(\mu+i\alpha)$$

There is one case in which we know all of these numbers. Suppose $V = V(\lambda)$ has highest weight λ and we take $\mu = \lambda$. Then $m_V(\lambda+i\alpha) = m_\lambda(\lambda+i\alpha) = 0$ for all $\alpha\in\Phi_+$ by definition of highest weight. So,

$$c = (\lambda,\lambda)+2(\lambda,\delta) = (\lambda+\delta,\lambda+\delta)-(\delta,\delta) = ||\lambda+\delta||^2 - ||\delta||^2$$

Therefore,

$$\begin{aligned} c-(\mu,\mu)-2(\mu,\delta) &= c-(\mu+\delta,\mu+\delta)+(\delta,\delta) \\ &= (\lambda+\delta,\lambda+\delta)-(\mu+\delta,\mu+\delta) \\ &= ||\lambda+\delta||^2 - ||\mu+\delta||^2 \end{aligned}$$

This gives the following theorem.

Theorem 24.3.1 (Freudenthal multiplicity formula)**.** *$m_\lambda(\lambda) = 1$ and for all $\mu < \lambda$ we have*

$$m_\lambda(\mu) = \frac{2\sum_{\alpha\in\Phi_+}\sum_{i=1}^{\infty}(\mu+i\alpha,\alpha)m_\lambda(\mu+i\alpha)}{||\lambda+\delta||^2-||\mu+\delta||^2}$$

This is a recursive formula for $m_\lambda(\mu)$ since the terms on the right involve only $m_\lambda(\nu)$ for $\mu<\nu\le\lambda$.

Example 24.3.2. Let $L = \mathfrak{sl}(3,F)$ and $\lambda = \lambda_1+\lambda_2 = \delta = \alpha_1+\alpha_2$. Then λ is a positive root and all roots in L have length 1. So,

$$||\lambda+\delta||^2 = 4||\lambda||^2 = 4$$

For $\mu = \alpha_2 = \lambda-\alpha_1$, $\mu+\delta = \alpha_1+2\alpha_2$ and $(\mu+\delta,\mu+\delta) = 3$. So, the formula gives

$$m_\lambda(\alpha_2) = 2(\lambda,\alpha_1)m_\lambda(\lambda) = 1$$

The case $\mu = \alpha_1$ also gives $m_\lambda(\alpha_1) = 1$ by symmetry.

For $\mu = 0$ we get

$$m_\lambda(0) = \frac{2((\alpha_1,\alpha_1)m_\lambda(\alpha_1)+(\alpha_2,\alpha_2)m_\lambda(\alpha_2)+(\lambda,\lambda)m_\lambda(\lambda))}{4-1} = 2$$

etc.