24.4. **Outline of the proof.** We want to prove the Weyl character formula

$$\omega(\delta)\mathrm{ch}_\lambda = \omega(\lambda+\delta)$$

In this form, both sides are alternating with highest weight $\lambda+\delta$ and the coefficient of $e(\lambda+\delta)$ is 1 on both sides.

Lemma 24.4.1. *The elements $\omega(\lambda)$, for λ strongly dominant, form a basis for $\mathbb{Z}[\Lambda]^-$, the additive group of alternating elements of $\mathbb{Z}[\Lambda]$.*

Proof. Every nonzero element of $\mathbb{Z}[\Lambda]$ has at least one highest weight λ which is dominant. But, if λ is dominant but not strongly dominant, then the coefficient of $e(\lambda)$ must be zero for any alternating element since $s_i\lambda = \lambda$ for some simple reflection s_i and $e(s_i\lambda) = e(\lambda)$ have coefficients of opposite sign (being alternating).

The lemma is now an easy induction. Choose a highest weight λ and subtract the appropriate multiple of $\omega(\lambda)$ to reduce the size of the support. □

Definition 24.4.2 (Dominance order)**.** If $\lambda, \mu \in \Lambda$ are integer weights, we say $\lambda > \mu$ if $\lambda - \mu$ is dominant.

Exercise 24.4.3. If λ, μ are strongly dominant weights with $\lambda > \mu$ then show that $||\lambda|| > ||\mu||$. (Since $(\mu, \lambda-\mu) \geq 0$.)

Lemma 24.4.4. *Let $\Delta : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]$ be the linear function given by*

$$\Delta\left(\sum n_\mu e(\mu)\right) = \sum n_\mu (\mu,\mu) e(\mu)$$

then

$$\Delta\left(\sum a_\lambda \omega(\lambda)\right) = \sum a_\lambda ||\lambda||^2 \omega(\lambda)$$

To prove the character formula it is enough to show that

$$\Delta(\omega(\delta)\mathrm{ch}_\lambda) = ||\lambda+\delta||^2 \omega(\delta)\mathrm{ch}_\lambda$$

The reason is that, $\omega(\delta)\mathrm{ch}_\lambda$ is alternating and has highest weight $\lambda+\delta$. The other weights that occur in $\omega(\delta)\mathrm{ch}_\lambda$ are

$$\mu + \sigma(\delta) < \lambda + \delta$$

since $\mu \leq \lambda$ and $\sigma(\delta) \leq \delta$.

24.4.1. *Formulas for $\omega(\delta)$.* Recall that $\delta = \sum_{\alpha>0} \frac{1}{2}\alpha$ where $\alpha > 0$ means $\alpha \in \Phi_+$. Thus

$$e(\delta) = \prod_{\alpha>0} e(\alpha/2)$$

We used the following formula in the proof of the dimension formula (Theorem 22.8.1) and we need it again.

Lemma 24.4.5.

$$\omega(\delta) = \prod_{\alpha>0}\left(e\left(\frac{\alpha}{2}\right) - e\left(\frac{-\alpha}{2}\right)\right) = \prod_{\alpha>0} e(\delta)(1-e(-\alpha))$$

Proof. Both sides are alternating with the same highest weight term $e(\delta) = \prod_{\alpha>0} e(\alpha/2)$. But δ is the minimal strongly dominant weight. □

Let $r = |\Phi_+|$ be the number of positive roots. Then

$$(-1)^r \omega(\delta) = \prod_{\alpha<0} \left(e\left(\frac{\alpha}{2}\right) - e\left(\frac{-\alpha}{2}\right)\right)$$

since each of the r terms on the RHS is the negative of the corresponding term in the equation above. Therefore,

$$(-1)^r \omega(\delta)^2 = \prod_{\alpha} \left(e\left(\frac{\alpha}{2}\right) - e\left(\frac{-\alpha}{2}\right)\right)$$

where the product is now over all roots α. Call this expression P. Since $\prod_\alpha e(\alpha/2) = e(\sum \alpha/2) = e(0) = 1$ we have

$$P = \prod_\alpha (e(\alpha) - 1)$$

The plan is now the following. To prove the Weyl character formula, we will take the equation that we used to prove the Freudenthal formula, multiply both sides by P, then differentiate using the operator d defined below.

24.5. **The derivation d.** Let d be the linear differential operator on $\mathbb{Z}[\Lambda]$ defined by

$$d \sum a_\mu e(\mu) = \sum a_\mu e(\mu)\, d\mu \in \mathbb{Z}[LL] \otimes dH^*$$

where $d\mu$ is a symbol which is linear in $\mu \in H^*$. More precisely, $d\mu$ is an element of dH^* which is a free H^* module generated by the single element d. In particular, we have

$$d(\mu + \nu) = d\mu + d\nu$$

Lemma 24.5.1. *$d : \mathbb{Z}[LL] \to \mathbb{Z}[LL] \otimes dH^*$ is a derivation.*

Proof. Let $A = \sum a_\mu e(\mu)$ and $B = \sum b_\nu e(\nu)$. Then

$$AB = \sum_{\mu,\nu} a_\mu b_\nu e(\mu + \nu)$$

So,

$$d(AB) = \sum_{\mu,\nu} a_\mu b_\nu e(\mu+\nu) d(\mu+\nu)$$

$$= \sum_{\mu,\nu} a_\mu b_\nu e(\mu) e(\nu) d\mu + \sum_{\mu,\nu} a_\mu b_\nu e(\mu) e(\nu) d\nu$$

$$= BdA + AdB$$

□

Example 24.5.2. Since $P = \prod_\alpha (e(\alpha) - 1)$ and $d(e(\alpha) - 1) = e(\alpha) d\alpha$, we have

$$dP = \sum_\alpha P_\alpha e(\alpha)\, d\alpha$$

where $P_\alpha = \prod_{\beta \neq \alpha} (e(\beta) - 1)$. Since $P = (-1)^r \omega(\delta)^2$ we also have

$$dP = (-1)^r 2\omega(\delta)\, d\omega(\delta)$$

Definition 24.5.3. Let $\Omega = \mathbb{Z}[\Lambda] \otimes dH^*$ considered as a left $\mathbb{Z}[\Lambda]$ module and let $(-,-) : \Omega \otimes \Omega \to \mathbb{C}[\Lambda]$ be given by

$$\left(\sum A_\mu d\mu, \sum B_\nu d\nu \right) = \sum_{\mu,\nu} A_\mu B_\nu (\mu, \nu)$$

Lemma 24.5.4. *For any $A, B \in \mathbb{Z}[\Lambda]$ we have*

$$\Delta(AB) = B\Delta A + A\Delta B + 2(dA, dB)$$

Proof. Since both sides are bilinear in A, B it suffices to consider the case $A = e(\mu), B = e(\nu)$. Then

$$\Delta(e(\mu)e(\nu)) = ||\mu + \nu||^2 e(\mu + \nu)$$

On the other side we have

$$(||\mu||^2 + ||\nu||^2 + 2(\mu, \nu)) e(\mu + \nu)$$

which are equal. □

24.6. **Proof of Weyl character formula.** Recall the formula:

$$c\, m_\lambda(\mu) = (\mu, \mu) m_\lambda(\mu) + \sum_{\alpha \in \Phi} \sum_{i=0}^{\infty} (\mu + i\alpha, \alpha) m_\lambda(\mu + i\alpha)$$

where $c = ||\lambda + \delta||^2 - ||\mu + \delta||^2$. Multiply by $e(\mu)$ and sum over all weights μ to get

$$\underbrace{c \sum_\mu m_\lambda(\mu) e(\mu)}_{c\,\mathrm{ch}_\lambda} = \underbrace{\sum_\mu ||\mu||^2 m_\lambda(\mu) e(\mu)}_{\Delta \mathrm{ch}_\lambda} + \sum_\mu \sum_\alpha \sum_{i \geq 0} (\mu + i\alpha, \alpha) m_\lambda(\mu + i\alpha) e(\mu)$$

The last term simplifies if we multiply by $P = P_\alpha(e(\alpha) - 1)$ since

$$(e(\alpha) - 1) \sum_\mu \sum_{i \geq 0} (\mu + i\alpha, \alpha) m_\lambda(\mu + i\alpha) e(\mu) = \sum_\mu (\mu, \alpha) m_\lambda(\mu) e(\mu + \alpha)$$

Therefore, P times the triple summation is

$$\sum_\mu \sum_\alpha (\mu, \alpha) P_a m_\lambda(\mu) e(\mu + \alpha) = (dP, d\mathrm{ch}_\lambda)$$

This gives:

$$cP\mathrm{ch}_\lambda = P\Delta \mathrm{ch}_\lambda + (dP, d\mathrm{ch}_\lambda)$$

Now expand $P = (-1)^r \omega(\delta)^2$.

$$(-1)^r c\omega(\delta)^2 \mathrm{ch}_\lambda = (-1)^r \omega(\delta)^2 \Delta \mathrm{ch}_\lambda + (-1)^r 2\omega(\delta)(d\omega(\delta), d\mathrm{ch}_\lambda)$$

Since $\mathbb{C}[\Lambda] \cong \mathbb{C}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]$ is an integral domain, we can cancel $(-1)^r\omega(\delta)$ from both sides to get

$$\begin{aligned} c\omega(\delta)\mathrm{ch}_\lambda &= \omega(\delta)\Delta\mathrm{ch}_\lambda + 2(d\omega(\delta), d\mathrm{ch}_\lambda) \\ &= \Delta(\omega(\delta)\mathrm{ch}_\lambda) - \mathrm{ch}_\lambda\Delta(\omega(\delta)) \end{aligned}$$

But,

$$c = ||\lambda+\delta||^2 - ||\delta||^2$$
$$\Delta\omega(\delta) = ||\delta||^2$$

So, we get

$$||\lambda+\delta||^2\omega(\delta)\mathrm{ch}_\lambda = \Delta(\omega(\delta)\mathrm{ch}_\lambda)$$

proving that

$$\omega(\delta)\mathrm{ch}_\lambda = \omega(\lambda+\delta)$$

as I explained earlier.

24.7. **Kostant's formula.** For any integer weight ν let $p(\nu)$ be the number of ways that ν can be written as a sum of positive roots $\alpha \in \Phi_+$. This is also equal to the number of ways that $-\nu$ can be written as a sum of negative roots. So,

$$\sum_\nu p(\nu)e(-\nu) = \prod_{\alpha>0} \frac{1}{1-e(-\alpha)}$$

Theorem 24.7.1.

$$m_\lambda(\mu) = \sum_{\sigma\in W} \mathrm{sgn}(\sigma) p(\sigma(\lambda+\delta) - (\mu+\delta))$$

Proof. Since $\omega(\delta) = e(\delta)\prod_{\alpha>0}(1-e(-\alpha))$, we have

$$\omega(\delta)^{-1} = e(-\delta)\prod_{\alpha>0}\frac{1}{1-e(-\alpha)} = \sum_\nu p(\nu)e(-\nu-\delta)$$

Therefore,

$$\mathrm{ch}_\lambda = \omega(\delta)^{-1}\omega(\lambda+\delta) = \sum_\nu p(\nu)e(-\nu-\delta)\sum_{\sigma\in W}\mathrm{sgn}(\sigma)e(\sigma(\lambda+\delta))$$

$$\sum_\mu m_\lambda(\mu)e(\mu) = \sum_{\sigma\in W}\sum_\nu \mathrm{sgn}(\sigma)p(\nu)e(\sigma(\lambda+\delta)-(\nu+\delta))$$

Thus $m_\lambda(\mu)$ is the coefficient of $e(\mu)$ on the right. But

$$\mu = \sigma(\lambda+\delta) - (\nu+\delta)$$

is equivalent to

$$\nu = \sigma(\lambda+\delta) - (\mu+\delta)$$

So, the coefficient of $e(\mu)$ is

$$m_\lambda(\mu) = \sum_{\sigma\in W} \mathrm{sgn}(\sigma) p(\sigma(\lambda+\delta) - (\mu+\delta))$$

□