In response to the request that writing intensive courses should have directions on how to write, I am posting these "Proof writing notes". This second note is about:

Counterexamples

In Homework 2 there are several questions which ask you to "Prove it or give a counterexample." The idea of the counterexample is based on the fact that most statements and theorems are "general" or "universal" in the sense that they state that the conclusion always holds. When the statement or theorem is of that kind then it takes only one counterexample to disprove it. The counterexample should be specific enough so that there is at least one example.

Preliminary: First check that the statement is general or universal.

Step 1 of the Counterexample. Specify the counterexample. It does not work to have a vague counterexample. The counterexample should be really simple.

Step 2. Explain which conclusion the counterexample fails to satisfy and why.

Step 3. Explain why your counterexample satisfies all of the conditions of the theorem.

Step 4. State the conclusion: The counterexample shows that the theorem is false.

Example 1

Definition $f(S) := \{f(s) | s \in S\}$

Problem Prove or find a counterexample: If $f : A \to B$ is a function and X, Y are subsets of A then $f(X \cap Y) = f(X) \cap f(Y)$.

Answer This statement is false. A counterexample is given by $A = B = \mathbb{Z}$, $f(x) = x^2$, $X = \{2\}, Y = \{-2\}$. Clearly X, Y are subsets of $\mathbb{Z}$ with empty intersection. So,

$$f(X \cap Y) = f(\emptyset) = \emptyset.$$

But, $f(X) = \{f(2)\} = \{4\} = f(Y)$. So,

$$f(X) \cap f(Y) = \{4\} \cap \{4\} = \{4\} \neq f(X \cap Y) = \emptyset$$

showing that the statement fails for this example.

Example 2

Definition Let A be any set. Then a function $f : A \to \mathbb{R}$ is called *nowhere zero* if $f(a) \neq 0$ for all $a \in A$. In other words, $0 \notin f(A)$.

Problem Prove or find a counterexample: If $f : A \to \mathbb{R}$ is a function which is bounded and nowhere zero then $1/f$ is bounded.

Answer A counterexample is given as follows. Let A be the open interval $A = (0, 2)$ and let $f(a) = a$. This function is bounded by 2 on A since the elements of A are bounded by 2 in absolute value. Also, f is nowhere zero on A since A does not contain 0. However, $1/f$ is unbounded since A contains $1/n$ for every positive integer n and

$$\frac{1}{f(1/n)} = \frac{1}{1/n} = n$$

which is unbounded. This counterexample shows that the statement is false: $1/f$ need not be bounded.