

After looking at your homework it seems that many of you need to understand the difference between:

Finding the answer *vs* Writing the proof

The key point is: Write it backwards! When you write a proof you start with the answer and work back to the question. In calculus this is called “checking your answer”. You verify that the answer that you got is correct. That step is called the “proof”. The first step is called the “derivation” of the answer. In all other math courses you need to derive the answer to show where it came from. In writing proofs, the derivation is not necessary. We just want to see the step where you carefully check the answer.

Example 1

Find the image of the function $f : (0, \infty) \rightarrow \mathbb{R}$ given by

$$f(x) = \frac{1}{x^2}$$

and prove that it is correct.

You find the answer in three steps.

Step 1. (the “usual method”) Figure out what the answer is. In this case you analyze the function using limits:

- 1) As x goes to zero, $f(x)$ goes to $+\infty$.
 - 2) As x goes to infinity, $f(x)$ goes to zero.
 - 3) The intermediate value theorem (IVT) implies that all values between 0 and ∞ are in the image. So, the image contains $\mathbb{R}_{>0} = (0, \infty)$.
 - 4) Since $x^2 > 0$ the fraction $f(x) = 1/x^2$ is always positive. Therefore the image is contained in $\mathbb{R}_{>0}$.
 - 5) Therefore, the image is equal to $\mathbb{R}_{>0}$.
- 3') (instead of IVT): For each y in the image (i.e., $y > 0$) you need to find an $x > 0$ so that $y = f(x)$. You take some scrap paper and write:

$$\begin{aligned} y &= 1/x^2 \\ x^2 &= 1/y \\ x &= \pm\sqrt{1/y} \end{aligned}$$

The positive answer is the correct one.

Step 2. (follow the outline of the appropriate proof method) Let $I = f((0, \infty))$ be the image of the function f . Let $\mathbb{R}_{>0} = \{y \in \mathbb{R} | y > 0\}$ be the set of positive real numbers. We want to show that $I = \mathbb{R}_{>0}$. To prove this we need to show $I \subseteq \mathbb{R}_{>0}$ and $\mathbb{R}_{>0} \subseteq I$.

To show that $I \subseteq \mathbb{R}_{>0}$ let $y \in I$. Then, for some $x \in (0, \infty)$,

$$y = f(x) = 1/x^2$$

This is clearly positive since $x^2 > 0$. Therefore, $y \in \mathbb{R}_{>0}$.

To show that $\mathbb{R}_{>0} \subseteq I$ let $y \in \mathbb{R}_{>0}$. Then we need to show that $y \in I$. This is the same as saying that $y = f(x)$ for some $x \in (0, \infty)$. Let

$$x = \sqrt{1/y}$$

Then $x^2 = 1/y$ and

$$f(x) = \frac{1}{x^2} = \frac{1}{1/y} = y$$

Therefore, $y \in I$. We showed that $I \subseteq \mathbb{R}_{>0}$ and $\mathbb{R}_{>0} \subseteq I$. Therefore, $I = \mathbb{R}_{>0}$.

Comment: The calculation that you do at the end of this proof is the derivation (3') backwards. Also, note that the logical structure of the above proof follows a standard pattern: It shows that two sets are equal by showing that they contain each other.

Step 3. (optional) Rewrite the proof to make it more to the point, avoid repetitions, and to sound better: I claim that the image of the function $f(x) = 1/x^2$ on the domain $\mathbb{R}_{>0}$ is the set $\mathbb{R}_{>0}$ of positive real numbers. Since the numerator 1 and denominator x^2 are both positive, the fraction $1/x^2$ is positive. Therefore, the image lies in $\mathbb{R}_{>0}$. Conversely, any $y > 0$ is in the image of f since $y = f(x)$ for $x = \sqrt{1/y}$. The key point is that, when $y > 0$, $1/y > 0$. So, $x = \sqrt{1/y}$ is a positive real number in the domain of f . The following calculation verifies the statement that $y = f(x)$ for this choice of x .

$$f(x) = \frac{1}{x^2} = \frac{1}{1/y} = y.$$

This proves the claim.

Comments: In the rewritten proof, I decided that the important point is that x and y are positive. So, I stressed in the proof that $x = \sqrt{1/y}$ is in the domain of the function.

Another point is that the limit and IVT derivation needs a lot of work to be a real proof. We need to prove that the IVT applies to the case at hand and determine what exactly it says. Also we need to verify the limit calculations in order to use the outline in Step 1 as a proof.

Example 2

Show that

$$x^2/4 + y^2 \geq xy$$

for all real numbers x and y .

Step 1. Figure out why it is true:

$$x^2/4 + y^2 \geq xy$$

$$x^2 + 4y^2 \geq 4xy$$

$$x^2 + 4y^2 - 4xy \geq 0$$

is true since $x^2 + 4y^2 - 4xy = (x - 2y)^2 \geq 0$.

Step 2. Write the derivation backwards to get the proof:

$$(x - 2y)^2 \geq 0$$

So,

$$x^2 + 4y^2 - 4xy \geq 0$$

Add $4xy$ to each side to get:

$$x^2 + 4y^2 \geq 4xy$$

Divide by 4 to get

$$x^2/4 + y^2 \geq xy$$

Step 3. Rewrite the proof. For any $x, y \in \mathbb{R}$, $(x - 2y)^2 \geq 0$ since the square of any real number is ≥ 0 . Expanding the product we get:

$$(x - 2y)^2 = x^2 + 4y^2 - 4xy \geq 0.$$

Add $4xy$ to each side to get:

$$x^2 + 4y^2 \geq 4xy.$$

Divide by 4 (multiply by $1/4$) to get the inequality that we wanted to prove.

Comments: In the rewritten proof, I corrected some flaws in Step 2. In general each line needs to be both justified and placed in proper context. Also, in the first line, the variables are crying out to be properly quantified. A typical complete line of a computational proof would look like:

Thus, $x = y = z$ since (blah, blah, blah).

But, $x \geq 0$ since otherwise (something terrible would happen).

The single word at the beginning puts the equation into context: Is it given? Does it follow? Does it contradict?