The coin problem seems to be more difficult than intended. So, I am writing these notes. First, I will go over the basic problem, the one I did in class.

BASIC COIN PROBLEM

You have n coins. One of them is heavier than the others. You have a scale with two trays which will tell which tray is heavier. How many weighings do you need to find the heavy coin?

Answer: The number of weighings needed is equal to the smallest positive integer k so that

$$n \leq 3^k.$$

The proof is by induction on n. If $n = 1$ you already know which coin it is. So, you need $k = 0$ weighings and $n = 1 \leq 3^0 = 1$ holds. If $n = 2$ or $n = 3$ then you put two coins on the scale and weigh them against each other. You either find that one is heavier or that they balance, in which case the one left off the scale is the heavy one. In all of these cases, you needed only $k = 1$ weighing and $2, 3 \leq 3^1 = 3$. So, the equation holds in these cases as well.

Proceeding by induction, suppose that $n \geq 4$ and the formula holds for all numbers from 1 to $n - 1$. Then we will show that k weighings is sufficient. It is certainly necessary since, for k weighings, there are only 3^k possible outcomes (3 for each weighing). Therefore, we will not be able to distinguish between more than 3^k possibilities with k weighings. So, $n \leq 3^k$. The thing we need to prove is that k is the smallest number satisfying this equation.

In the first weighing, we should divide the coins into three piles as evenly as possible. To do this, divide n by 3 and round up. Since $n \leq 3^k$, $n/3 \leq 3^{k-1}$. Round the fraction $n/3$ up to the next whole number and call it m. Then $m \leq 3^{k-1}$ and each pile has m or $m - 1$ coins. There must be at least two piles with the same number of coins. Weigh these against each other. If they balance, the heavy coin is in the third pile. If they do not balance, you know which side has the heavy coin. In any case you will have narrowed your search to one pile having m or $m - 1$ coins. Since $m \leq 3^{k-1}$ and $m < n$ we know by induction on n that you can find the heavy coin in $k - 1$ more weighings or less. So, the total number of weighings is k or less. (The reason it might be less is that $k - 1$ may not be the smallest number in the equation $m - 1 \leq 3^{k-1}$. For example, if $n = 11$ then $k = 3$ but $m - 1 = 3 \leq 3^1$. So, it might happen that you find the coin in just 2 weighings.)

2-WAY COIN PROBLEM

Now suppose that you have n coins, a scale plus an extra standard coin. (So, you actually have $n+1$ coins.) One of the coins has the wrong weight. (It is either too heavy or too light) How many weighings do you need to determine which coin is different and whether it is too heavy or too light? [Hint: A counting argument shows that the minimum number of weighings should be the smallest integer k so that

$$2n \leq 3^k$$

Try to find an algorithm which will give you the answer in that many weighings.] Prove your answer using induction. [Prove that your algorithm will work using strong induction.]

This is your homework question. So, I will just outline the procedure.

You divide the coins into three piles. Leave as many as you can off the scale so that you can find the coin in $k-1$ weighings if the scale balances. Then put the rest on the scale using the good coin to make an even number if needed. Then you will have two sets of coins. You will have H number of coins on the heavy side (one of them might be heavy) and L on the light side (one of these might be light) and the key equation is:

$$H + L \leq 3^{k-1}$$

You then need to find the bad coin in $k-1$ weighings. You can't use induction yet. You need to do another weighing. You put a of the H coins on each tray and b of the L coins on each tray and weigh them. (Find a and b.) As a result of this second weighing, you will have a coins which might be heavy and b coins which might be light. If you chose a, b correctly you have

$$a + b \leq 3^{k-2}$$

Then, by induction, you know that you can find the coin in $k-2$ weighings. The reason you can use induction only after the second weighing is that the situation changes after the first weighing but it does not change after the second weighing (only numbers get smaller).

To make this a real proof you need to state the induction hypothesis. What is the statement that you are proving by induction? It is true that the numbers H, L are either equal or differ by 1. But if you use this fact, you will create a big headache for yourself because, by the rules of induction, you will need to prove that the two numbers a, b are either equal or differ by 1.

There is another more sophisticated solution which involves using markers and Scotch tape: If the scales don't balance, mark the possibly heavy coins blue and the possibly light coins red and Scotch tape together pairs of red and blue coins. Then you are reduced to the same problem with a smaller number. Whenever there is an unpaired coin you leave it off the scale in the next weighing. Tape two unmarked coins together to use as the new good coin.