

1. [10 points] Find a counterexample to the following statement.

If f and g are functions from \mathbb{R} to \mathbb{R} with the property that $f + g$ is bounded and $f(x) \geq 0$ for all x in \mathbb{R} then g is bounded.

Answer: Let $f(x) = x^2$ and $g(x) = -x^2$. This is a counterexample since $x^2 \geq 0$ and $|g(x)| = x^2$ is unbounded.

Don't forget: Write in complete sentences. Write a few extra obvious words to make equations into sentences.

2. [10 points] Write the following statement in logical notation, find its negation, then write the negation in English. Use P for the set of all primes and O for the set of odd integers.

For every positive integer n , if $n + 7$ is a sum of two primes then n is odd.

- a) This statement, in logical notation, is

$$(\forall n \in \mathbb{Z}_{>0})[(\exists p_1, p_2 \in P)n + 7 = p_1 + p_2] \Rightarrow (\exists k \in \mathbb{Z})n = 2k + 1.$$

- b) Since this has the form $(\forall n)p \Rightarrow q$ the negation is $(\exists n)p \wedge \neg q$ or

$$(\exists n \in \mathbb{Z}_{>0})[(\exists p_1, p_2 \in P)n + 7 = p_1 + p_2] \wedge (\exists k \in \mathbb{Z})n = 2k.$$

- c) The negation in English is:

There is a positive even integer n so that $n + 7$ is the sum of two primes.

- d) (Not required) Examples are given by $n = 2, 4, 6, 8, 12, 14, \dots$. These are counterexamples of the original statement. All such counterexamples have the form $n = p - 5$ where p is a prime larger than 5. Then $n + 7 = 2 + p$ is the sum of the two primes 2 and p .

3. [10 points] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by the formula $f(x) = |x| + |1 - x|$. Prove that the minimum value of f is 1.

Answer: Since $f(0) = |0| + |1| = 1$ the value 1 is attained by the function. So, we just have to prove that $f(x) \geq 1$ for all $x \in \mathbb{R}$. There are three cases. Either $x < 0$, $0 \leq x \leq 1$ or $x > 1$. [Note that it is obvious that this list gives all possible values of x .]

Case 1. $x < 0$. Then $x = -z$ for some $z > 0$. So,

$$f(x) = |-z| + |1 + z| = z + 1 + z = 2z + 1$$

This is > 1 since $z > 0$.

Case 2. $0 \leq x \leq 1$. In this case, x and $1 - x$ are both ≥ 0 . So,

$$f(x) = |x| + |1 - x| = x + 1 - x = 1 \geq 1.$$

Case 3. $x > 1$. In this case $x > 0$ but $1 - x < 0$. Also, $2x > 2(1) = 2$. So,

$$f(x) = |x| + |1 - x| = x + x - 1 = 2x - 1 > 2 - 1 = 1.$$

Therefore, $f(x) \geq 1$ in all cases.