Math 23b **Quiz 1 Answers to Practice b** **Sept, 2007**

Rules: Closed book. No calculators. Write all answers in your exam booklet. Write clearly, using complete sentences.

1. [10 points] Find a counterexample to the following statement.

   If $f : A \to B$ is a function from set A to set B and C, D are subsets of A so that $f(C) = f(D)$ then $C = D$.

   Answer 1: (a specific counterexample) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let C, D be the one-element subsets of $A = \mathbb{R}$ given by $C = \{2\}, D = \{-2\}$. Then $C \neq D$ but $f(C) = f(D) = \{4\}$.

   Answer 2: (an abstract counterexample) Let $A = \{a, b\}$ where a, b are distinct. Let $B = \{c\}$. Let $f : A \to B$ be the function given by $f(a) = f(b) = c$. Let C, D be the subsets of A given by $C = \{a\}, D = \{b\}$. Then $f(C) = f(D) = B$ but $C \neq D$.

   Note that in the counterexample, you need to say what is what in terms of the original language of the question. In Answer 1, A, C, D are explicitly given and it is understood that $B = \mathbb{R}$ since it says $f : \mathbb{R} \to \mathbb{R}$. Also, there is a subtle point: In mathematics, different letters can stand for the same quantity. So, it is important to write that a, b are not equal.

2. [5 points] Suppose that you are going to prove the following statement by contradiction. How would the proof start? (You do not have to finish the proof!)

   If $|a| < 4$ then for all $x \in \mathbb{R}$, $x^2 + ax + 4 \geq 0$.

   (In this statement, a is a real number.)

   Answer: Suppose the statement is not true. Then there exist real numbers a and x so that $|a| < 4$ and $x^2 + ax + 4 < 0$. We will show that this leads to a contradiction.

   Analysis of the question. To get to this answer, I did the following scratch work. The letter a is understood to be universally quantified. So, the statement is

   $$(\forall a \in \mathbb{R})[|a| < 4 \Rightarrow (\forall x \in \mathbb{R})x^2 + ax + 4 \geq 0]$$

   The negation is

   $$(\exists a \in \mathbb{R})[|a| < 4 \wedge (\exists x \in \mathbb{R})x^2 + ax + 4 < 0]$$

   Since this is an "and" statement, we can pull all quantifiers to the beginning:

   $$(\exists a, x \in \mathbb{R})|a| < 4 \wedge x^2 + ax + 4 < 0.$$

   In English, this gives the second sentence in the answer. The first sentence, or something like it, is necessary. The third sentence makes it clear that I gave a complete answer to the question that was asked, i.e., the remainder of the proof consists of logical deductions leading to a contradiction. Also, without the third sentence, the paragraph would not be complete.

3. [5 points] If x, y are positive real numbers so that $xy \geq 20$ then prove, using the contrapositive, that either $x \geq 4$ or $y > 5$.

Answer: We will prove the contrapositive: *If* $x < 4$ *and* $y \leq 5$ *then* $xy < 20$. This follows in two steps:

1) $x < 4$ implies that $xy < 4y$ since $y > 0$.
2) $y \leq 5$ implies that $4y \leq 4 \cdot 5 = 20$. So, $xy < 4y \leq 20$ which implies $xy < 20$ as claimed.

4. [10 points] Find the image of the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{2}{1+x^2}$$

and prove it.

Answer: The image is the half-open interval $(0, 2]$. To prove this, first I will show that the image is contained in this set. Then I will show that each point in this interval is in the image.

First of all, the fraction $2/(1 + x^2)$ is always positive since $x^2 \geq 0$. Second, since $x^2 \geq 0$ we get

$$2 + 2x^2 \geq 2.$$

Dividing both sides by the positive number $1 + x^2$, we get

$$2 \geq \frac{2}{1+x^2}.$$

So, $0 < f(x) \leq 2$ for all real x.

Now suppose that $y \in (0, 2]$. Then $2 - y \geq 0$ and $y > 0$. So

$$\frac{2-y}{y} \geq 0$$

which means that we can take the square root. So, let

$$x = \sqrt{\frac{2-y}{y}}.$$

Then, the following calculation shows that $y = f(x)$.

$$1 + x^2 = 1 + \frac{2-y}{y} = \frac{2}{y}.$$

So,

$$f(x) = \frac{2}{1+x^2} = \frac{2}{2/y} = \frac{2y}{2} = y.$$

Since y was an arbitrary element of $(0, 2]$ this shows that $(0, 2]$ is contained in the image. So, $(0, 2]$ is the image of f.