MATH 23B NOTES 2009

13. The real numbers

We are following the axiomatic method which consists of the following.

(1) We have a list of properties satisfied by the real numbers. They are called Axioms because *all other properties of real number follow from these*. There is only one number system satisfying the axioms.
(2) We study the consequences of the axioms in detail. The world community now agrees that the number system specified by these axioms is what we think of as the "real numbers."
(3) We make a mathematical model for the real numbers and prove that the Axioms hold on our model. (See Appendix A.)

The Axioms are listed on page 16. They are all very ordinary and boring (such as associativity and commutativity of addition and multiplication). But there is one very interesting axiom called the **Completeness Axiom**.

13.1. **The Completeness Axiom.** You want an axiom that says there are no "holes" in the real number line. For example $\sqrt{2}$. We proved earlier that there is no rational number whose square is 2. The ancient Greeks who believed in number (the Pythagoreans) concluded that $\sqrt{2}$ *does not exist*. However, the diagonal of a unit square has length exactly $\sqrt{2}$. So, even 2000 years ago, some people realized that $\sqrt{2}$ is real.

The challenge is the show that $\sqrt{2}$ is a *number*. As a number, it is not so easy to construct. The best we can do is say that there exists some infinite decimal expansion:

$$1.41421356237309504880168872420969807856967187537694807317667973799073247846210703888$$

But this infinite decimal is actually a sequence of fractions:

$$1,\quad \frac{14}{10},\quad \frac{141}{100},\quad \frac{1414}{1000},\quad \frac{14142}{10000},\quad \frac{141421}{100000},\quad \frac{1414213}{1000000},\quad \frac{14142135}{10000000},\quad \frac{141421356}{100000000},\quad \frac{141421356}{1000000000}$$

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which get closer and closer to the number that we have in our mind. This particular sequence or fractions has the property that it is non-decreasing. (When we get to the first decimal which is 0 then two consecutive terms are equal.)

Problem: Find the formula for these fractions.

Now, let's look at the Completeness Axiom and see if it does the job it is designed to do, namely to fill in all the "holes" in the number line.

Axiom 13.1 (Completeness Axiom for real numbers)**.** *Every nonempty subset of real numbers with an upper bound has a least upper bound.*

Remember that an **upper bound** for a subset $S \subseteq \mathbb{R}$ is a real number α so that $x \leq \alpha$ for all $x \in S$. The **least upper bound** or **supremum** of S is defined to be the smallest upper bound:

$$\sup S = \beta$$

means β is an upper bound for S and $\beta \leq \alpha$ for any other upper bound for S. The Completeness Axiom says that such a real number β exists.
The Completeness Axiom is also equivalent to the

Greatest Lower Bound Property: *Every nonempty subset S of $\mathbb{R}$ with a lower bound has a greatest lower bound.* This is denoted

$$\inf S$$

Equivalent mean that the two properties imply each other. (You just replace numbers with their negatives to go from one to the other.)

Example 13.2.
(1) *What is the supremum and infimum of the half open interval $S = [0, 1)$? This shows that* $\sup S$ *and* $\inf S$ *are sometimes in S and sometimes not in S.*
(2) *Find an upper bound for the sequence of fractions converging to $\sqrt{2}$ and prove that it is an upper bound.*
(3) *Show that the set of rational numbers does not satisfy the Completeness Axiom.*
(4) *Find a set S of rational numbers for which $\sqrt{2}$ is a least upper bound.*

13.2. **Limits and monotone convergence.**

Definition 13.3. *A sequence* $\langle a\rangle$ *of real numbers* $a_1, a_2, \cdots$ *has* **limit** $L \in \mathbb{R}$ *(or* **converges** *to* L*) if*

$$(\forall\epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)|a_n - L| < \epsilon$$

which reads: For all $\epsilon > 0$ *there is some number* N *so that for every* $n > N$, $|a_n = L| < \epsilon$.

In different words, this says that the sequence of numbers eventually gets close to L and stays there. According to this definition, a sequence cannot have two limits. For example:

$$a_n = (-1)^n(1 + \frac{1}{n})$$

This is the sequence

$$-2\ ,\ \frac{3}{2}\ ,\ -\frac{4}{3},\ \frac{5}{4}\ ,\ -\frac{6}{5}\ ,\ \frac{7}{6}\ ,\ -\frac{8}{7}\ ,\ \frac{9}{8}\ ,\ \cdots$$

which intuitively "converges to both 1 and -1" since the numbers get closer and closer to 1 and -1. However, they do not stay close to either number. It would not be true to say that

$$|a_n - 1| < \epsilon$$

for all large n since half the time it is not no matter how large n is.

Theorem 13.4. *A sequence can have at most one limit.*

Proof. This is a proof by contradiction. Suppose that L_1, L_2 are limits for the same sequence $\langle a\rangle$. Let

$$\epsilon = \frac{1}{2}|L_1 - L_2|$$

Then the definition (applied twice) says that there exist $N_1, N_2 \in \mathbb{N}$ so that for all $n > N_1$, $|a_n - L_1| < \epsilon$ and for all $n > N_2$, $|a_n - L_2| < \epsilon$. Take $n = N_1 + N_2$. This is larger than both numbers. So, by the triangle inequality,

$$2\epsilon = |L_1 - L_2| \leq |a_n - L_1| + |a_n - L_2| < \epsilon + \epsilon = 2\epsilon$$

This is a contradiction. So, the theorem must be true. □

Example 13.5.
(1) *Find other sequences which do not converge for other reasons. How many different reasons can you find?*
(2) *Prove that* $a_n = 1/n$ *converges to 0.*