MATH 23B, LECTURE NOTES

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Contents

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1. Numbers, sets and functions

1.1. **Inequalities.**

1.1.1. *Quadratic equation.* The first topic was the quadratic equation. We started with the problem:

Find two (real) numbers whose sum is s and whose product is p.

First, we convert this into an equation. Let x, y be two numbers whose sum is $x + y = s$ and whose product is $xy = p$. Then, solving the first equation for y and plugging the answer in the second equation we get

$$y = s - x$$
$$p = xy = x(s - x) = sx - x^2$$

or:

$$x^2 - sx + p = 0$$

And the solution is:

$$x = \frac{s \pm \sqrt{s^2 - 4p}}{2}$$

But negative numbers do not have square roots. So: there are three cases:

(1) $s^2 - 4p < 0$. In this case, there are no (real) solutions to the problem.
(2) $s^2 = 4p = 0$. Then there is only one solution:
$$x = y = \frac{s}{2}$$
(3) $s^2 - 4p > 0$. Then there are the two solutions given by the equation above.

Next, I converted this into a proof question:

Suppose that x, y are two real numbers so that $xy = 6$ and $x > 2$. Then show that $y < 3$.

The solution is to take the inequality $x > 2$ and multiply both sides by 3. Then we get:

$$3x > 6$$

And we are given that $xy = 6$, so

$$3x > xy$$

We want to show (wts) that $3 > y$. So, we need to divide by x. The rule that we need to do this is:

If $a > b$ and $c > 0$ then $ac > bc$

We apply this to $a = 3x, b = xy, c = 1/x$. We are given $x > 2$. So, x and $1/x$ are positive. This gives the final answer

$$3 > y.$$

□

Worksheet problem: *Given that x, y are real numbers whose product is $xy = 10$ and $x > 5$, prove that $y < 2$.*

Another worksheet problem was: *Find the set of all real numbers x so that*

$$x^2 - 7x + 10 < 0$$

Hint: First solve the equation $x^2 - 7x + 10 = 0$.

The solution involves more about the quadratic equation which I somehow assumed that everyone knew. This is that the polynomial factors:

$$x^2 - 7x + 10 = (x - 2)(x - 5)$$

In order for this to be negative, one of the factors must be negative and one must be positive: So, either

(1) $x - 2 > 0$ and $x - 5 < 0$ or
(2) $x - 2 < 0$ and $x - 5 > 0$.

But the second case is not possible. Why? So, the first case is true:

$$x - 2 > 0, \text{ so } x > 2$$

and

$$x - 5 < 0, \text{ so } x < 5$$

So, the solution of the inequality is:

$$2 < x < 5$$

So, the solution set S is:

$$S = \{x \in \mathbb{R} \mid 2 < x < 5\}$$

which is also called the open interval $(2, 5)$. [This is, unfortunately, the same notation as for the point in the xy-plane with coordinates $x = 2, y = 5$. So, to be really precise, we need to write the words "open inverval"]

1.1.2. *Definition and Axioms.* In the proof I used one of the properties of inequalities. I explained that inequalities are given by a definition and three axioms.

Definition 1.1. If a, b are real numbers then $a > b$ means $a - b$ is positive. We write $a - b \in P$ where P is the set of positive real numbers.

The concept of positivity satisfies three axioms:

P1 *The sum of two positive numbers is positive.*

$$x, y \in P \Rightarrow x + y \in P$$

P2 *The product of two positive numbers is positive.*

$$x, y \in P \Rightarrow xy \in P$$

P3 *Every real number is either positive, zero or negative. (And these three possibilities are mutually exclusive.*

$$x \in P \text{ or } x = 0 \text{ or } -x \in P$$

All other properties of inequalities follow from these.

Lemma 1.2. *If $a < b$ and $c > 0$ then $ac < bc$.*

Proof. We are given that $a < b$. By definition, this means $b - a$ is positive. We are also given that c is positive. So, by Axiom $P2$ we have:

$$c(b - a) = cb - ca \in P$$

By definition, this mean

$$cb > ca$$

which is what we wanted to prove (although written slightly differently). □

Note: this proof has the hidden understanding that $a < b$ is the same as $b > a$. Also, $ca = ac$ and $cb = bc$. In a truly rigorous text we would put these tedious details. But, as I said in class, I will assume the axioms of arithmetic such as the commutativity of addition and multiplication and the distributive property.

Very similar to this, we have the problem on the worksheet:

Use $P1$ to prove the transitivity *property of inequalities which says:*
If $a > b$ and $b > c$ then $a > c$.

The solution is: $a > b, b > c$ mean that $a - b$ and $b - c$ are positive. So, by $P1$,

$$(a - b) + (b - c) = a - c \in P$$

So, $a > c$ □

Theorem 1.3. *If $0 < a < b$ then $0 < a^2 < ab < b^2$ and $0 < \sqrt{a} < \sqrt{b}$*

Proof. Using the lemma with $c = a$, we get

$$a^2 < ab$$

Since b is also positive, we get

$$ab < b^2$$

This show the first statement. The second statement is proved by "contradiction".

As Sherlock Holmes said: *"When you have eliminated the impossible, whatever remains, however improbable, must be the truth."*

We are trying to show that $\sqrt{a} < \sqrt{b}$ which is the same as $\sqrt{b} - \sqrt{a} \in P$. And we know, by Axiom P3, that this number is either positive, zero or negative. The three possibilities are:

(1) $\sqrt{b} - \sqrt{a} > 0$ (This is what we want to show.)
(2) $\sqrt{b} - \sqrt{a} = 0$, or equivalently, $\sqrt{b} = \sqrt{a}$.
(3) $\sqrt{b} - \sqrt{a} < 0$ or equivalently, $\sqrt{b} < \sqrt{a}$.

To prove that the first case is true, we need to eliminate the other two cases. To do this you assume that one of the other cases is true. This is called *proof by contradiction*. You *assume* that that case we want to prove is not true and it is one of the other possibilities which holds.

Suppose *by contradiction* that either the second case or third case is true. In the second case, $\sqrt{b} = \sqrt{a}$. Squaring both sides we get $b = a$ which contradicts the assumption that $a < b$. So, this is impossible.

Suppose that the third case is true. Then $\sqrt{b} < \sqrt{a}$. We just proved that this implies that

$$(\sqrt{b})^2 < (\sqrt{a})^2$$

In other words, $b < a$, which contradicts the assumption $b > a$. So, the third case is also impossible.

Therefore, we conclude that remaining possibility must be true:

$$\sqrt{b} > \sqrt{a}$$

The contradiction proves that the theorem holds. □

In the book, cases (2) and (3) are combined to

$$\sqrt{b} \leq \sqrt{a}$$

The complete statement and proof of this approach is as follows.

We are given theat $0 < a < b$ and we want to show $\sqrt{a} < \sqrt{b}$.

Proof. The proof is by contradiction. Suppose that $\sqrt{a} \geq \sqrt{b}$. Since square roots are nonnegative by definition, when we square both sides we get $a \geq b$ which contradicts the assumption that $a > b$. □

This proof uses the following fact.

Lemma 1.4. *If $x \geq y \geq 0$ then $x^2 \geq y^2$.*

Proof. The complete proof is done by *Cases*. We are given that $x > y$ <u>or</u> $x = y$ and either $y > 0$ <u>or</u> $y = 0$. In each case, we know that the conclusion holds:

(1) If $x > y$ and $y > 0$ then we proved that $x^2 > y^2$.
(2) If $x = y$ then $x^2 = y^2$ so $x^2 \geq y^2$ is true.
(3) If $y = 0$ then $x^2 \geq 0 = y^2$.

□

Problem: Give an example to show that the following statement is NOT true. (Mathematical statements are true if they are always true. Just one exception make the statement false.)

If $a > b$ then $a^2 > b^2$.

1.2. **Inequalities and sets.** On the first day I proved the rule that, if $a < b$ and $c > 0$ then $ac < bc$. I want to consider the other two cases: $c = 0$ and $c < 0$.

If $c = 0$ then $ac = 0 = bc$. Also, if $a = b$ then $ac = bc$ no matter what c is. So, we conclude that:

Theorem 1.5. *If $a < b$ and $c \geq 0$ then $ac \leq bc$.*

Proof. There are two cases. $c > 0$ and $c = 0$. In the first case we already proved that $ac < bc$ so, a fortiori, $ac \leq bc$. In the second case, $ac = 0 = bc$. So the conclusion holds in all cases. □

Theorem 1.6. *If $a < b$ and $c < 0$ then $ac > bc$.*

Multiplying by a negative number reverses the inequality. An example is: $2 < 3$ but $-2 > -3$. But an example is not a proof.

Proof. The product of the two positive numbers $b-a$ and $-c$ is positive:

$$-c(b - a) = ac - bc > 0$$

So, $bc < ac$. □

I will talk more about inequalities and sets. We will go over the "triangle inequality" and the "AGM" inequality.

Theorem 1.7 (AGM inequality)**.** *If x, y are nonnegative real numbers then the geometric mean $\sqrt{xy}$ is $\le$ the arithmetic mean $\frac{x+y}{2}$:*

$$\sqrt{xy} \le \frac{x+y}{2}$$

and equality holds only when $x = y$. In general, if x, y are any two real numbers then

$$xy \le \left(\frac{x+y}{2}\right)^2$$

and, again, equality holds only when $x = y$.

The proof is in the book. I want to explain how to use this.

Problem: Show, *without using calculus* that the largest rectangle with perimeter equal to 12 is a 3×3 square.

Answer: A rectangle with width x and height y has area $A = xy$ and perimeter $P = 2x + 2y$. If the perimeter is $P = 12$ then $x + y = 6$. So, the AGM inequality says:

$$A = xy \le \left(\frac{x+y}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 3^2 = 9$$

So, the area is at most 9 and it is equal to 9 only if $x = y = 3$ which means the rectangle is a square.

Definition 1.8. The *absolute value* of a real number x is defined to be:

$$|x| := \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is the distance from x to 0.

There is another equation for absolute value which does both cases at the same time:

$$|x| = \sqrt{x^2}$$

Problem: Prove that $|x||y| = |xy|$.

Theorem 1.9 (Triangle inequality)**.** *If x, y are real numbers then*

$$|x + y| \le |x| + |y|$$

Proof. When you want to prove an equation of inequality with absolute values, you first need to get rid of the absolute value sign somehow. The idea is to square both sides. But this is WRONG!! You cannot square both sides of an equation that you do not know is true. Instead we do a proof by contradiction.

In a standard proof by contradiction, we are saying: Either the statement is true or it is false. To prove the first we assume the second and get a contradiction.

Suppose by contradiction that

$$|x + y| > |x| + |y|$$

Now, we have an assumption. Since both sides are nonnegative we can square them to get:

$$x^2 + 2xy + y^2 > (|x| + |y|)^2 = |x|^2 + 2|x||y| + |y|^2 = x^2 + 2|xy| + y^2$$

Subtracting $x^2 + y^2$ from both sides we get:

$$2xy > 2|xy|$$

which is a contradiction since $|xy| \geq xy$ by definition of absolute value. This contradiction proves the theorem. □

Next, I will do the last unexplained problem from the first day:

Find the set S of all positive real numbers x so that

$$\left|\frac{x-4}{x}\right| \leq 2$$

Hint: first get rid of the absolute value signs.

To eliminate the absolute value sign, we need to know that $|z| \leq 2$ means $-2 \leq z \leq 2$.

For the case at hand we get:

$$-2 \leq \frac{x-4}{x} \leq 2$$

Let's pause to consider the logic of *solving an equation* vs *proving and equation*

(1) To solve an equation, we *assume that it is true*, i.e., that we have a solution x. Then we deduce what x must be.
(2) To prove an equation, we *assume that it is not true*, then we get a contradiction proving that "not true" is not an option.

Multiply by x which is given to be positive, we get

$$-2x \leq x - 4 \leq 2x$$

Subtract x from all three terms to get

$$-3x \leq -4 \leq x$$

This is two statements:

$$-4 \leq x \text{ and } -3x \leq -4$$

We are looking for the set of all positive x satisfying both of these inequalities. The first is true for any positive x. For the second, we reverse the signs to get:

$$3x \geq 4$$

Divide by 3 to get

$$x \geq \frac{4}{3}$$

The logic is the following: We started with the assumption that $x \in S$ and we ended with the conclusion that $x \geq 4/3$. So, what this proves is that S is contained in the set

$$A = [4/3, \infty) = \{x \in \mathbb{R} \,|\, x \geq 4/3\}$$

To prove that this is the answer, we also need to prove that $A \subseteq S$. Let me explain the set theory and then come back to this.

Definition 1.10. A set S is *contained in* a set T if every element of S is an element of T. Two sets S and T are *equal* if they are contained in each other.

Example: If exactly the same students are registered in Math 1 and Econ 1 then the two class, *as sets of students*, are equal.

Getting back to the problem, to show that $A \subseteq S$ we take an element $x \in A$ (a letter representing an arbitrary element of A) Then we need to show that $x \in S$, i.e. that it satisfies the inequality.

The assumption is $x \in A$ which means $x \geq 4/3$. Multiply both sides by $3/x$ to get

$$3 \geq \frac{4}{x} > 0$$

Subtract 1 from everything:

$$2 \geq \frac{4}{x} - 1 > -1$$

So,

$$\left| \frac{4}{x} - 1 \right| \leq 2$$

But

$$\left| \frac{4}{x} - 1 \right| = \left| \frac{4}{x} - \frac{x}{x} \right| = \left| \frac{4-x}{x} \right| = \left| \frac{x-4}{x} \right|$$

So, $x \in S$. This proves $A \subseteq S$. Since we already know that $S \subseteq A$ we conclude that $A = S$.

1.3. **Worksheet:** Find the set S of all real numbers x so that

$$\left|\frac{x+4}{x}\right| < 1$$

and prove that your answer is the correct set. (It is understood that $x \neq 0$.)

Hint: First, get rid of the absolute value sign. Then consider separately the two possibilities: $x > 0$ and $x < 0$. The solution has three steps.

Step 1: Find the set of all positive x satisfying the inequality above. Show that there are no positive solutions. This set is empty!

Step 2: Find the set of all negative x satisfying the inequality.

It helps to write $x = -y$ in the second case.

Step 3: Prove that your answer is correct. If you did it correctly, what you have so far is that the solution set S is contained in your answer which I will call A. So, you need to prove that A is contained in S. To do this, take any element $x \in A$ and show that it is in S.

second problem

Prove, by contradiction: *If x, y are real numbers with product $xy = 10$ and $|x| > 2$ then $|y| \leq 5$*

Start with the negation of the conclusion. Then you have three assumptions. What are they?

Next, using the theorem from class arrive at a contradiction.

Finally, you need to write the conclusion, namely that the contradiction proves the result.

If you finish this you can start doing HW1.

1.4. **Homework 1.** HW1 is due next Thursday. The rules for the homework are: You can work in groups but you have to hand in your own version of the answer in your own words. Those words are very important since this is a writing course! Don't try to be wordy but you must write in *complete sentences*. My wife (also a math teacher) says: Write the answer so that your peers can understand it.

(1) Find the set S of all x satisfying the inequality
$$x^2 - 4x + 5 \leq 10$$
Hint: First make the inequality and equality and solve it. Are your solutions to the equality in the set S or not?

(2) Prove that your answer to the first problem is correct. (I haven't explained how to do this yet.)

(3) Prove by contradiction that if $xy \leq 9$ and $x > 3$ then $y < 3$.

(4) Find an example to show that the following statement is NOT true: *If* $xy \geq 9$ *and* $x < 3$ *then* $y > 3$. [Write your answer in words: "Let ... This show that ..."]

(5) Find the set T of all real numbers x so that
$$\left| \frac{x}{x-2} \right| < 4$$
[Hint: Divide into two cases: $x > 2$ and $x < 2$.] Show that T is equal to the set you name as your answer.

(6) You own land along a straight river. You have 400 meters of fence. Prove using AGM (without using calculus!) that the largest rectangular area that you can fence in from three sides using your fencing material (using the river as your fourth side) is 100×200. [Hint: Take a $x \times 2y$ rectangle where $2y$ is the distance along the river.]

1.5. **equality of sets.** (This part I will probably not get to in class. But you need it for your homework. I will explain it on Monday.)

Definition 1.11. A set S is *contained in* a set T if every element of S is an element of T. Two sets S and T are *equal* if they are contained in each other.

Here is an example.

Show that the set S of solutions of the inequality

$$x^2 - 4x < 0$$

is the open interval $S = (0, 4) = \{x \in \mathbb{R} \,|\, 0 < x < 4\}$.

The problem is to show that the two sets S (the solution set) and $T = (0, 4)$ are equal. To do this we show that $S \subseteq T$ and $T \subseteq S$.

(1) First, we show $T \subseteq S$. This mean that the elements of T are in S. So, suppose $x \in T$. Then $x < 4$. So $x - 4 < 0$. Multiplying both sides by $x > 0$ we get

$$x^2 - 4x = x(x - 4) < x(0) = 0$$

So, $x \in S$ which proves $T \subseteq S$.

(2) Next, take any element $x \in S$. This means x is some solution of the inequality above. So:

$$x(x - 4) < 0$$

We need to prove that $x \in (0, 4)$. But our assumption $x(x-4) < 0$ implies that one of the factors is positive and one is negative. There are two possibilities:

(a) $x > 0$ and $x - 4 < 0$ which implies $x < 4$. This is what we want to show.

(b) $x < 0$ and $x - 4 > 0$ which implies $x > 4$ is positive. This is a contradiction.

Since the second case is impossible, the first is true. So $x \in T$. This shows $S \subseteq T$

Since $S \subseteq T$ and $T \subseteq S$ we conclude that $S = T$.

2. More about sets

First you start with a big set U called the "universe" consisting of all of the things your are considering. For example, we could take

$$U = \mathbb{R} = \text{ the set of all real numbers}$$

W can talk about Union, intersection, complement, difference of subsets of U.

(1) *Union* The union of two sets $A \cup B$ is the set of all $x \in U$ so that either $x \in A$ or $x \in B$. Normally student say "and" instead of "or" when talk about the union.
(2) *Intersection* The intersection of two sets $A \cap B$ is the set of all $x \in U$ so that $x \in A$ and $x \in B$.
(3) *Complement* The complement of A is the set of all $x \in U$ so that x is not in A ($x \notin A$). The complement of A is denoted A^c.
(4) *Difference* The difference $A - B$ is the set of all $x \in A$ so that x is not in B.

$$A - B = A \cap B^c$$

Problem: Show that for any two sets A and B we have:

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$$

You can draw a picture, but a picture is not a proof.

Proof. To show that two sets are equal you take an arbitrary element of one set and show that it is in the other set.

(1) Take any $x \in (A \cup B) - (A \cap B)$ then we need to show that x is in the other set.

We are given that x is in $A \cup B$ but not in $A \cap B$. So, x is either in A or B but not not both. Therefore, either x is in A but not in B ($x \in A - B$) or x is in B but not in A ($x \in B - A$). Therefore, x is in the union of these two sets. So

$$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$$

(2) Take any $x \in (A - B) \cup (B - A)$. Then either $x \in A - B$ or $x \in B - A$. In the first case, $x \in A - B$, we get that x is in A and therefore in the union but x is not in the intersection since it is not in B. The secon case is similar. So $x \in A \cup B - A \cap B$. This shows that

$$(A \cup B) - (A \cap B) \supseteq (A - B) \cup (B - A)$$

Since the two sets are contained in each other, they are equal by definition of equality of sets. $\square$

One interesting example of a set is the set of all subsets of a set A. This is called the *power set* of A and denoted $\mathcal{P}(A)$. For example, if $A = \{a, b\}$ then A has 4 subsets: $A, \{a\}, \{b\}, \emptyset$. In words: the entire set, the set containing only a, the set containing only b and the empty set. This means the power set is:

$$\mathcal{P}(A) = \{A, \{a\}, \{b\}, \emptyset\}$$

Theorem 2.1. *If A has n elements then the power set has 2^n elements.*

The reason for this is that every element of the power set can be represented by a string of n 0's and 1's where the 1's indicate which elements are in the set. For example: if $A = \{a, b, c, d, e\}$ then 00101 represents the subset $\{c, e\}$ since there are 1's in the third and 5th entry. This subset is an element of $\mathcal{P}(A)$.

On the other hand, there are exactly 2^n binary sequences of length n. We will prove this by induction later. For now, we should just take it as "obvious." Therefore there are 2^n subsets.

Definition 2.2. The *Cartesian product* of two set $S \times T$ is the set of all ordered pairs (s, t) where $s \in S$ and $t \in T$. For example,

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \,|\, x, y \in \mathbb{R}\}$$

This is the xy-plane.

Notation:

$\mathbb{R}$ is the set of all real number

$\mathbb{Z}$ is the set of all integers

$\mathbb{N}$ is the set of all natural numbers. This not well-defined. "well defined" means there is a precise membership criterion. The question "Is 0 a natural number?" does not have a consistent answer. In our book, 0 is not a natural number. If we agree on that then $\mathbb{N}$ is well-defined.

$\mathbb{Q}$ is the set of rational numbers:

$$\frac{a}{b}$$

where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. (If a, b have no common factors then the fraction is called "reduced")

2.1. **Worksheet.** (a) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Hint: start by letting x denote an element of the first set and show that x is in the second.

(a1) If $x \in A \cap (B \cup C)$ then what do you know about x? Is x in A? in B? in C?

(a2) Going the other way, you start with an element of the second set and show that it is in the first set. There are two cases. What are they?

(b) Let $A = \{a, b, c\}$

(b1) List all the subsets of A.

(b2) Write binary expressions for each of these subsets.

(c1) Draw the set $\mathbb{Z} \times \mathbb{N}$ in the plane.

(c2) What is $\mathbb{Z} \times \mathbb{N} \cap \mathbb{N} \times \mathbb{Z}$?

(c3) If A, B are subsets of $\mathbb{R}$ then prove that

$$(A \times \mathbb{R}) \cap (\mathbb{R} \times B) = A \times B$$

2.2. **functions.**

Definition 2.3. A function or *mapping* f from a set A to a set B is written:

$$f : A \to B$$

This is defined to be a rule which assigns one element of B to every element of A. A is called the *domain* and B is called the *target*.

Example 2.4. Let $A = \{a, b, c\}$ and $B = \mathbb{Z}$ and define the function $f : A \to \mathbb{Z}$ by $f(a) = 5, f(b) = 2, f(c) = 5$. Notice that this function takes the value 5 twice.

If $f(a) = b$ then we say:
b is the *value* of the function f at a and
b is the *image* of a under f.
Instead of naming the target set B we can say "B-valued function"
For example if we say "f is a real-valued function on A" we mean:

$$f : A \to \mathbb{R}$$

In the example above, f is an integer valued function on A.

Definition 2.5. The *image* of a function f is the set of all values that it takes:

$$image(f) = f(A) = \{f(a) \,|\, a \in A\}$$

First a simple example to explain the language. Take the function

$$g : [2, \infty) \to \mathbb{R}$$

given by

$$g(x) = x^2 + 5$$

Here the domain is the half closed interval

$$[2, \infty) = \{x \in \mathbb{R} \,|\, x \geq 2\}$$

The image of this function is the set of all real numbers ≥ 5:

$$g([2, \infty)) = [9, \infty)$$

Problem: *Prove that this is the image (without using calculus)*

Answer: [*with commentary in italics*] To show that these sets are equal, we take an element of each set and show it is in the other.

1) Take any element of the image $g([2, \infty))$. By definition this is $g(x)$ where $x \geq 2$. Since $x \geq 2$ we have $x^2 \geq 4$. Adding 5 to both sides we get $x^2 + 5 \geq 9$. But this is $g(x)$:

$$g(x) = x^2 + 5 \geq 9$$

This proves that the image of g is contained in the set $[9, \infty)$.

2) Now take any $y \in [9, \infty)$. *(You can use any letter. I use y to remind me that this is supposed to be $y = g(x)$.)*

Then we need to show that it is in the image of the function g. *This means we need to solve the equation*

$$y = x^2 + 5$$

and show that our solution is in the domain. The equation is easy to solve:

$$x = \sqrt{y - 5}$$

We just need to show that this x is in the given domain of g.

We are given $y \geq 9$, so $y - 5 \geq 4$ and therefore $\sqrt{y-5} \geq 2$. So $x = \sqrt{y-5}$ is in the domain of g and

$$g(x) = x^2 + 5 = \left(\sqrt{y-5}\right)^2 + 5 = (y - 5) + 5 = y$$

Therefore, y is in the image of g. Since y is an arbitrary element of $[9, \infty)$, this shows that $[9, \infty)$ is contained in the image of g.

Since the image contains this set and this set is contained in the image we conclude that the two sets are equal:

$$image(g) = g([2, \infty)) = [9, \infty)$$

[*"im" is an abbeviations for "imaginary" so don't use "im" for image.*]

Definition 2.6. The *graph* of a function $f : A \to B$ is the subset of the Cartesian product $A \times B$ consisting of all ordered pairs $(a, f(a))$ where $a \in A$.

Definition 2.7. A subset S of $\mathbb{R}$ is called *bounded* if there is a positive real number $M \in \mathbb{R}$ so that $|x| \leq M$ for all $x \in S$. A real valued function on any set $f : A \to \mathbb{R}$ is *bounded* if its image is bounded.

For example, the functions $[0, \infty) \to \mathbb{R}$ given by $f(x) = x^3 - x$, $g(x) = \sqrt{x}$, $h(x) = |x|$ are unbounded but the functions $f(x) = \sin x$ and $g(x) = 1/(1 + x)$ are bounded on the domain $[0, \infty)$.

Problem: Show that $g(x) = 1/(1 + x)$ is bounded on $[0, \infty)$.

Claim: $0 < \frac{1}{1+x} \leq 1$ for all $x \geq 0$.

First, note that this claim proves that the function $g(x)$ is bounded. So, we just need to prove this Claim.

The proof of the Claim is by contradiction. Suppose that $\frac{1}{1+x} > 1$ for some $x \geq 0$. Then we will get a contradiction which will prove the claim. If $\frac{1}{1+x} > 1$ then, multiplying both sides by $1 + x > 0$ we get the inequality $1 > 1 + x$. Subtract 1 from both sides to get $0 > x$ which contradicts the assumption that $x \geq 0$. So, we are done.

2.3. **Worksheet 4.** (a) Let f be the real valued function on the set of positive integers given by the equation

$$f(k) = \frac{k}{k+1}$$

(a1) What are the domain and target of this function?

(a2) Find the image of the function.

(a3) Hopefully, your answer to (a2) is a set. Prove that your set is the image of f.

(a4) Prove that the function f is bounded. What is the bound M? [Use your answer to (a3) to give a short answer to (a4).]

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be the function:

$$g(x) = 1 + 2|x|$$

(b1) Find the image of this function.

(b2) Find the inverse image of the point 5. Find the inverse image of the set $(0, 5)$. [See definition and example on next page.]

Definition 2.8. If $f : A \to B$ is a mapping (=function) and $C \subseteq B$ then the *inverse image* of C is defined to be the set of all $a \in A$ so that $f(a) \in C$:

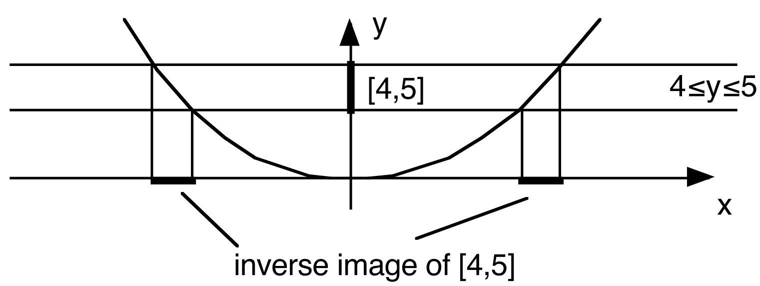
$$f^{-1}(C) = \{a \in A \mid f(a) \in C\}$$

If $b \in B$ is one element then the *inverse image* of b is the set of all $a \in A$ so that $f(a) = b$.

For example, if $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = x^2$, then the inverse image of 4 is the set $f^{-1}(4) = \{2, -2\}$ and the inverse image of the closed interval $[4, 5]$ is the set

$$f^{-1}[4,5] = [-\sqrt{5}, -2] \cup [2, \sqrt{5}]$$

This can be visualized by figure below. The target set is traditionally drawn on the y-axis. So, the subset $[4, 5]$ is the the segment on the y-axis. You draw horizontal lines at 4 and 5 and the inverse image is the projection to the x-axis of the portion of the graph between these lines. Giving a rigorous proof of concepts given by drawings is very difficult. Normally we write "as illustrated in the figure" or "as suggested by the figure" then we still need to write a proof.



In the book, inverse image is written I_f for example:

$$I_f(4) = \{-2, 2\}$$

However, f^{-1} is standard notation even though it can be confused with inverse function. So, you need to write words: "the inverse image $f^{-1}(C)$ of C"

2.4. **increasing decreasing functions are one-to-one.** Today, I will explain the concept of "one-to-one" or "1-1" or "injective" function using a more familiar concept of increasing and decreasing functions.

Definition 2.9. Suppose $f : S \to T$ is a function where S and T are subsets of the real numbers. We say that f is an *increasing* function if $f(a) < f(b)$ for any $a < b$ in the domain S. We say f is *decreasing* if $f(a) > f(b)$ for all $a < b$.

Here are two examples. Let $S = \mathbb{N}$ (the set of positive integers) and $T = \mathbb{Z}$ (the set of all integers) and $f : \mathbb{N} \to \mathbb{Z}$ is given by

$$f(n) = n^2 + n$$

n	1	2	3	4	5	6	7	...
$f(n)$	2	6	12	20	30	42	56	...

This function is increasing. I will prove it later.

Let $g : \mathbb{Z} \to \mathbb{R}$ be given by

$$g(n) = n^3 - n$$

n	...	-3	-2	-1	0	1	2	3	...
$g(n)$	...	-24	-6	0	0	0	6	24	...

This function is neither increasing nor decreasing. The reason is very simple. The numbers $0, 1$ are in the domain $\mathbb{Z}$ and $0 < 1$ but it is not true that $g(0) < g(1)$ since they are equal:

$$g(0) = 0 = g(1)$$

Another simple example: The function

$$h : \mathbb{R} \to \mathbb{R}$$

given by $h(x) = |x|$ is not increasing or decreasing since

$$h(-1) = 1 = h(1)$$

To prove that a function h is NOT increasing, you need to find two particular numbers (not letters) a, b in the domain so that $a < b$ but $h(a)$ is not less than $h(b)$. In other words, $h(a) \geq h(b)$. So, when I pick the particular numbers $a = 0, b = 1$ for the case of the function g and calculate that $g(0) = g(1)$ then that proves that g is not increasing. It also proves that g is not decreasing since if it were then $g(0)$ would be greater than $g(1)$. (This is the subjunctive case exploring the hypothetical case that g is decreasing.)

It is easy to show that a function is NOT increasing or NOT decreasing since you just need two particular numbers and two calculations. To show that a function IS increasing (or decreasing) you have to use letters representing all possible pairs of numbers in the domain.

Here is the proof that $f(n) = n^2 + n$ is increasing on the domain $\mathbb{N}$.

Proof. Suppose that a, b are any two positive integers and $a < b$. Then we will show that $f(a) < f(b)$. This will prove that f is increasing.

Since $0 < a < b$ we know that $a^2 < b^2$. Using the *addition of inequalities* rule proved below we get

$$a^2 + a < b^2 + b$$

showing that f is increasing on its domain. □

In the future, we can use the *addition of inequalities* rule:

Theorem 2.10. *If $a < b$ and $x < y$ then $a + x < b + y$*

Proof. Adding x to both sides of the first inequality and b to both sides of the second inequality we get:

$$a + x < b + x < b + y$$

□

Definition 2.11. A function $f : A \to B$ is called *injective* or *one-to-one* or 1-1 if for any $a \neq b$ in the domain A, $f(a) \neq f(b)$.

Note: Every function takes one x to one y. An injective function takes two xs to two ys.

Theorem 2.12. *If a function $f : \mathbb{R} \to \mathbb{R}$ is increasing (or decreasing) then it is 1-1.*

Proof. Choose letters representing two different ("distinct") arguments, then show that you get two different values. Let a, b be two distinct real numbers. Then either $a < b$ or $a > b$. We may assume by symmetry that $a < b$. Since f is increasing $f(a) < f(b)$. In particular, $f(a) \neq f(b)$. Therefore, f is 1-1. □

The beauty of this type of mathematics is that we have a very short but completely rigorous proof written in a professional style. (The italicized comment is pedigogical, not part of the proof.)

Problem: Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - 2x$ is not 1-1.

This is easy: Take $0, 2$. These are two distinct elements of the domain but $f(0) = 0 = f(2)$ So the function is not 1-1.

Note that to show that a function is NOT 1-1 you take two particular numbers and show they have the same image. To show that a function is 1-1, you take letters representing two distinct elements of the domain.

2.5. **Worksheet 5.** (a) Don't look at the notes. Write the definition of an increasing function.

(a1) Give a new example of an increasing function (different from the ones that we discussed). [The idea is to find a very simple example. In mathematics we value originality and clarity not complexity.]

(a2) Give a new example of a function which is not increasing. [Here is an example. You need to find a different one. Let $A = [0, 1]$, $B = \mathbb{R}$, $f : [0, 1] \to \mathbb{R}$ is given by $f(x) = 5$ for all x. This is not increasing since $0 < 1$ in the domain but $f(0) = 5 = f(1)$.]

(b) Let $A = [1, 4]$. Let $g : A \to \mathbb{R}$ be the function given by

$$g(x) = 2x^2 + \sqrt{x}$$

Show that g is increasing without using calculus. [Let $1 \leq a < b \leq 4$ and prove that $g(a) < g(b)$.]

What is the inverse image of 0? [Are there any numbers in the interval $[1, 4]$ so that $g(x) = 0$?]

(c) Let $B = [1, 3]$ and $h : B \to \mathbb{R}$ the function given by

$$h(x) = x + \frac{6}{x}$$

Show that h is not decreasing. [Find two specific numbers $a < b$ in the domain and calculate $h(a), h(b)$.]

Is h 1-1? [Does it always take two xs to two ys or are there two xs which give the same y?]

2.6. **Homework 2.** This homework is due next Thursday, Feb 4. Make sure to write complete sentences for all your answers. Write the conclusion for each problem in a sentence at the end of each answer. [For example, "This shows that the function h is increasing."]

(1) Let C be the set of all students in our class. (To make this well-defined, these are the students enrolled at time $T = 1$pm on Jan 28, 2010 in Math 23b at Brandeis) Let U be the set of all undergraduates at Brandeis (more precisely, people who are enrolled as undergraduate students at the university at time T). Let A be the set of all $x \in U$ so that x is a senior at time T. In words, what is $(U - A) \cup C$? Are you a member of this set? What is the complement of this set (in U)?

What is $(C - A) \cap U$. Are you a member of this set? Can you write these sets with fewer symbols?

(2) Prove that $(A \cup B) \cap (C \cap D) = (A \cap C \cap D) \cup (B \cap C \cap D)$.

(3) Find the power set of the empty set.

(4) Let A be the three point set with elements a, b, c. Let $f : A \to \mathbb{Z}$ be the function given by $f(a) = 1, f(b) = 5, f(c) = 5$.

Find the graph of f.

What is the image of f?

What is the inverse image of 5?

Is this function 1-1? Prove it.

(5) Let $B = \{a, b\}$ and suppose that $g : B \to \mathbb{R}$ is the function $g(a) = 2, g(b) = 3$. Then it does not make sense to ask if this function is increasing or decreasing. Why not?

(6) Explain why any real valued function on a finite set is bounded. [First, explain what these words mean in symbols: This means $f : A \to B$ where ....]

(7) Find a function $f : \mathbb{R} \to \mathbb{R}$ which is increasing and bounded and prove it.