

**4.5. theory of induction.** The theory behind strong induction is easy to explain. It is based on the following property of positive integers.

**Axiom 4.2.** *Every nonempty set of positive integers  $S$  contains a least element.*

In other words: *If there is a positive integer  $n$  satisfying property  $P(n)$  then there exists a smallest positive integer  $n$  satisfying  $P(n)$ .*

We call this the **well-ordered** property of the set  $\mathbb{N}$ . The real numbers don't have this property since, e.g., there is no smallest real number. The set of nonnegative integers is also well-ordered.

Here is an example of the use of this property which also explains the logic behind strong induction.

**Theorem 4.3.** *There is no rational number  $n/m$  whose square is 2*

In other words there do not exist positive integers  $n, m$  so that

$$\frac{n^2}{m^2} = 2$$

The book says “ $\sqrt{2}$  is not a rational number” which states that there exists a real number whose square is 2 and that that real number is not a rational number.

*Proof.* Suppose by contradiction that  $2 = n^2/m^2$  for some positive integers  $n$  and  $m$ . This gives:

$$n^2 = 2m^2$$

$P(n)$ :  $n^2$  is twice the square of another number.

By the well-ordered property of  $\mathbb{N}$ , there is a smallest positive integer  $n$  with this property. (We say  $n$  is *minimal* with this property.)

Since  $n, m$  are positive integers, this equation implies two things:

- (1)  $n > m$  (since  $n^2 = 2m^2 = m^2 + m^2 > m^2$ )
- (2)  $n$  is even (if  $n$  were odd,  $n^2$  would be odd)

Therefore,  $n = 2k$  for some positive integer  $k$ . But then

$$n^2 = 4k^2 = 2m^2 \Rightarrow 2k^2 = m^2.$$

So,  $m^2$  is twice the square of another number. So,  $P(m)$  is true. But  $m < n$ . This contradicts the minimality of  $n$ . Therefore, there is no positive integer  $n$  with this property which proves the theorem.  $\square$

This was a strong induction argument in disguise. When we say  $n$  is the smallest positive integer with property  $P(n)$  we are saying that  $P(n)$  does not hold for all positive integers less than  $n$ . So, we are proving  $\neg P(n)$  by strong induction. And there was *no base case*.

**Theorem 4.4.** *A finite set  $A$  with  $n$  elements has exactly  $2^n$  subsets.*

In this example, I will do the proof twice, once using strong induction and the second time using the well-ordered property.

*Proof.* First take the case  $n = 0$ . So,  $A = \emptyset$ . The empty set has no proper subset. So  $\emptyset$  is the only subset of  $A$  and  $\mathcal{P}(A) = \{\emptyset\}$  has one element and  $1 = 2^0$  showing that the theorem holds for  $n = 0$ .

Suppose by induction that  $n \geq 1$  and the theorem is true for all nonnegative integers less than  $n$ . Let  $A$  be a set with  $n$  elements. Since  $n \geq 1$ ,  $A$  is nonempty. So, it has an element  $a \in A$ . If we delete this element we get  $B = A - \{a\}$  which is a set with  $n - 1$  elements. By induction,  $B$  has exactly  $2^{n-1}$  subsets.

For every subset  $S$  of  $A$ ,  $S \cap B$  is a subset of  $B$ . This is a 2-1 correspondence between subsets of  $A$  and subsets of  $B$  since, for every subset  $T \subseteq B$ , there are two subsets of  $A$  which correspond to  $T$ , namely  $T$  and  $T \cup \{a\}$ . So,  $A$  has twice as many subsets as  $B$ . So,  $A$  has

$$2 \cdot 2^{n-1} = 2^n$$

subsets proving that the theorem holds for all  $n \geq 0$ .  $\square$

Now I want to explain this proof in terms of the well-ordered property.

We know the theorem holds for  $n = 0$  (by the base case in the proof above). Suppose by contradiction that the theorem is not true. Then there is a finite set  $A$  so that  $A$  has  $n$  elements but the number of subsets is not equal to  $2^n$ . We also know  $n$  is positive since the theorem is true for  $n = 0$ . By the well-ordered property there is a smallest positive integer  $n$  for which there is such a counterexample. "Smallest"  $n$  means that the theorem holds for all sets with fewer than  $n$  elements. Take  $a \in A$ ,  $B = A - \{a\}$ . Then  $B$  has  $2^{n-1}$  subsets. So,  $A$  has  $2^n$  subsets (by the proof above). This contradicts the assumption that  $A$  does not satisfy the theorem. This contradiction proves the theorem for all finite sets  $A$ .

There is famous joke showing a false use of strong induction:

*Prove by induction that all horses have the same color.*

If you have one horse then it has the same color as itself. So, the theorem holds for  $n = 1$ . Suppose that you have  $n$  horses and suppose by induction that the statement holds for  $n - 1$ . Remove one horse. Then the remaining horses have the same color by induction. Put it back and remove a different horse. Again, the remaining horses have the same color. So, they all have the same color. QED.

4.6. **worksheet.** (1) Show using the strong induction/well-ordered property that

$$1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1$$

1a) First take the case  $n = 0$  and show it is true.

1b) Prove the statement ( $P(n)$ ) assuming it is true for all smaller numbers (in particular for  $n - 1$ ).

1c) What is the wording/logic if you are using strong induction? What is the logic if you are using the well-ordered property?

1) When  $n = 0$  the LHS is equal to 1 (Note that there are  $n + 1$  terms on the left side and the first term is  $2^0 = 1$ .) The RHS is equal to  $2^1 - 1 = 2 - 1 = 1$ . So, the equation holds for  $n = 0$ .

1b) Now suppose by induction that  $n \geq 1$  and the equation holds for all nonnegative numbers  $< n$ . Then

$$1 + 2 + 4 + 8 + \cdots + 2^n = 2^n - 1 + 2^n = 2 \cdot 2^n - 1 = 2^{n+1} - 1.$$

So, the equation holds for  $n$  and therefore, by induction it holds for all  $n \geq 0$ .

1c) The wording I gave is for strong induction. For the well-ordered property, I would have said: Suppose that the equation fails for some  $n$ . Then, by the well-ordered property, there is a smallest  $n$  for which it fails. But this implies that the equation holds for all smaller  $n$ . (Insert calculation above.) This gives a contradiction to the assumption that the equation does not hold. So, the equation must hold for all  $n$ .

(2) Show using the well-ordered property that there is no rational number  $n/m$  whose square is 3.

2a) Suppose that there are positive numbers  $n, m$  so that  $n^2/m^2 = 3$ . Express this as a property of the number  $n$ .

2b) Take the smallest  $n$  with this property and get a contradiction.

**4.7. Homework 4.** This homework is due on the Thursday after the break. Quiz 1 will be on the Wednesday after the break covering everything up through simple induction. More details will be posted. I will explain briefly the answers to HW4 in class on Monday and we will also review for the quiz.

In each problem start your answer by stating the assumptions and end it with a conclusion stating the result. In between, make sure each step is justified and explained.

(1) Prove by induction on  $n$  that

$$\sum_{i=1}^n 3i(i-1) = n^3 - n$$

(2) Write out the answers to the second problem in today's worksheet and hand it in as homework. (Show using the well-ordered property that there is no rational number  $n/m$  whose square is 3.)

(3) Take the sum:

$$\sum_{k=1}^n 5 \cdot 3^k$$

a) What is this sum for  $n = 1, 2, 3$ ?

b) Find a formula for the sum and prove it by induction.

(4) Find the number of squares (with integer side lengths) inside a  $3 \times n$  square and prove it using induction. For example, a  $3 \times 3$  square has 14 squares: 9 little squares small 4 two-by-two squares and one big square. (Note: The same formula does not work for all  $n$ .)

(5) You have  $3^n$  coins and one of them is heavier than the others. You also have a scale. Show that you can find the heavy coin in exactly  $n$  weighings.

(6) [3.49(b)] Determine for which positive integers  $n$  the following inequality holds and prove it by induction.

$$2^n \geq (n+1)^2$$

(7) ( $L$ -tiling) Show that  $R_n$  has an  $L$ -tiling if  $n$  is a power of 2. Look at the  $L$ -tiling problem in the book (p. 61). There is a complicated inductive solution for tiling  $R_n$  for arbitrary  $n$ . ( $R_n$  is the bottom right shape on p.61.) However, when  $n$  is a power of 2 ( $n = 2^k$ ), there is an easy simple induction proof. Your job is to find this proof and explain it.

(8) [3.19] Prove by induction on  $n$  that

$$(\forall n \in \mathbb{N})(\forall x, y \in \mathbb{R}) x < y \Rightarrow x^{2^k-1} < y^{2^k-1}$$