MATH 23B, LECTURE NOTES

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Contents

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1. Numbers, sets and functions

1.1. Inequalities.

1.1.1. *Quadratic equation.* The first topic was the quadratic equation. We started with the problem:

Find two (real) numbers whose sum is s and whose product is p.

First, we convert this into an equation. Let x, y be two numbers whose sum is $x + y = s$ and whose product is $xy = p$. Then, solving the first equation for y and plugging the answer in the second equation we get

$$y = s - x$$
$$p = xy = x(s - x) = sx - x^2$$

or:

$$x^2 - sx + p = 0$$

And the solution is:

$$x = \frac{s \pm \sqrt{s^2 - 4p}}{2}$$

But negative numbers do not have square roots. So: there are three cases:

(1) $s^2 - 4p < 0$. In this case, there are no (real) solutions to the problem.
(2) $s^2 = 4p = 0$. Then there is only one solution:
$$x = y = \frac{s}{2}$$
(3) $s^2 - 4p > 0$. Then there are the two solutions given by the equation above.

Next, I converted this into a proof question:

Suppose that x, y are two real numbers so that $xy = 6$ and $x > 2$. Then show that $y < 3$.

The solution is to take the inequality $x > 2$ and multiply both sides by 3. Then we get:

$$3x > 6$$

And we are given that $xy = 6$, so

$$3x > xy$$

We want to show (wts) that $3 > y$. So, we need to divide by x. The rule that we need to do this is:

If $a > b$ and $c > 0$ then $ac > bc$

We apply this to $a = 3x, b = xy, c = 1/x$. We are given $x > 2$. So, x and $1/x$ are positive. This gives the final answer

$$3 > y.$$

□

Worksheet problem: *Given that x, y are real numbers whose product is $xy = 10$ and $x > 5$, prove that $y < 2$.*

Another worksheet problem was: *Find the set of all real numbers x so that*

$$x^2 - 7x + 10 < 0$$

Hint: First solve the equation $x^2 - 7x + 10 = 0$.

The solution involves more about the quadratic equation which I somehow assumed that everyone knew. This is that the polynomial factors:

$$x^2 - 7x + 10 = (x - 2)(x - 5)$$

In order for this to be negative, one of the factors must be negative and one must be positive: So, either

(1) $x - 2 > 0$ and $x - 5 < 0$ or
(2) $x - 2 < 0$ and $x - 5 > 0$.

But the second case is not possible. Why? So, the first case is true:

$$x - 2 > 0, \text{ so } x > 2$$

and

$$x - 5 < 0, \text{ so } x < 5$$

So, the solution of the inequality is:

$$2 < x < 5$$

So, the solution set S is:

$$S = \{x \in \mathbb{R} \,|\, 2 < x < 5\}$$

which is also called the open interval $(2, 5)$. [This is, unfortunately, the same notation as for the point in the xy-plane with coordinates $x = 2, y = 5$. So, to be really precise, we need to write the words "open inverval"]

1.1.2. *Definition and Axioms.* In the proof I used one of the properties of inequalities. I explained that inequalities are given by a definition and three axioms.

Definition 1.1. If a, b are real numbers then $a > b$ means $a - b$ is positive. We write $a - b \in P$ where P is the set of positive real numbers.

The concept of positivity satisfies three axioms:

P1 *The sum of two positive numbers is positive.*

$$x, y \in P \Rightarrow x + y \in P$$

P2 *The product of two positive numbers is positive.*

$$x, y \in P \Rightarrow xy \in P$$

P3 *Every real number is either positive, zero or negative. (And these three possibilities are mutually exclusive.*

$$x \in P \text{ or } x = 0 \text{ or } -x \in P$$

All other properties of inequalities follow from these.

Lemma 1.2. *If $a < b$ and $c > 0$ then $ac < bc$.*

Proof. We are given that $a < b$. By definition, this means $b - a$ is positive. We are also given that c is positive. So, by Axiom $P2$ we have:

$$c(b - a) = cb - ca \in P$$

By definition, this mean

$$cb > ca$$

which is what we wanted to prove (although written slightly differently). □

Note: this proof has the hidden understanding that $a < b$ is the same as $b > a$. Also, $ca = ac$ and $cb = bc$. In a truly rigorous text we would put these tedious details. But, as I said in class, I will assume the axioms of arithmetic such as the commutativity of addition and multiplication and the distributive property.

Very similar to this, we have the problem on the worksheet:

Use $P1$ to prove the transitivity *property of inequalities which says:*
If $a > b$ and $b > c$ then $a > c$.

The solution is: $a > b, b > c$ mean that $a - b$ and $b - c$ are positive. So, by $P1$,

$$(a - b) + (b - c) = a - c \in P$$

So, $a > c$ □

Theorem 1.3. *If $0 < a < b$ then $0 < a^2 < ab < b^2$ and $0 < \sqrt{a} < \sqrt{b}$*

Proof. Using the lemma with $c = a$, we get

$$a^2 < ab$$

Since b is also positive, we get

$$ab < b^2$$

This show the first statement. The second statement is proved by "contradiction".

As Sherlock Holmes said: *"When you have eliminated the impossible, whatever remains, however improbable, must be the truth."*

We are trying to show that $\sqrt{a} < \sqrt{b}$ which is the same as $\sqrt{b} - \sqrt{a} \in P$. And we know, by Axiom P3, that this number is either positive, zero or negative. The three possibilities are:

(1) $\sqrt{b} - \sqrt{a} > 0$ (This is what we want to show.)
(2) $\sqrt{b} - \sqrt{a} = 0$, or equivalently, $\sqrt{b} = \sqrt{a}$.
(3) $\sqrt{b} - \sqrt{a} < 0$ or equivalently, $\sqrt{b} < \sqrt{a}$.

To prove that the first case is true, we need to eliminate the other two cases. To do this you assume that one of the other cases is true. This is called *proof by contradiction*. You *assume* that that case we want to prove is not true and it is one of the other possibilities which holds.

Suppose *by contradiction* that either the second case or third case is true. In the second case, $\sqrt{b} = \sqrt{a}$. Squaring both sides we get $b = a$ which contradicts the assumption that $a < b$. So, this is impossible.

Suppose that the third case is true. Then $\sqrt{b} < \sqrt{a}$. We just proved that this implies that

$$(\sqrt{b})^2 < (\sqrt{a})^2$$

In other words, $b < a$, which contradicts the assumption $b > a$. So, the third case is also impossible.

Therefore, we conclude that remaining possibility must be true:

$$\sqrt{b} > \sqrt{a}$$

The contradiction proves that the theorem holds. □

In the book, cases (2) and (3) are combined to

$$\sqrt{b} \leq \sqrt{a}$$

The complete statement and proof of this approach is as follows.

We are given theat $0 < a < b$ and we want to show $\sqrt{a} < \sqrt{b}$.

Proof. The proof is by contradiction. Suppose that $\sqrt{a} \geq \sqrt{b}$. Since square roots are nonnegative by definition, when we square both sides we get $a \geq b$ which contradicts the assumption that $a > b$. □

This proof uses the following fact.

Lemma 1.4. *If $x \geq y \geq 0$ then $x^2 \geq y^2$.*

Proof. The complete proof is done by *Cases*. We are given that $x > y$ or $x = y$ and either $y > 0$ or $y = 0$. In each case, we know that the conclusion holds:

(1) If $x > y$ and $y > 0$ then we proved that $x^2 > y^2$.
(2) If $x = y$ then $x^2 = y^2$ so $x^2 \geq y^2$ is true.
(3) If $y = 0$ then $x^2 \geq 0 = y^2$.

□

Problem: Give an example to show that the following statement is NOT true. (Mathematical statements are true if they are always true. Just one exception make the statement false.)

If $a > b$ then $a^2 > b^2$.

1.2. **Inequalities and sets.** On the first day I proved the rule that, if $a < b$ and $c > 0$ then $ac < bc$. I want to consider the other two cases: $c = 0$ and $c < 0$.

If $c = 0$ then $ac = 0 = bc$. Also, if $a = b$ then $ac = bc$ no matter what c is. So, we conclude that:

Theorem 1.5. *If $a < b$ and $c \geq 0$ then $ac \leq bc$.*

Proof. There are two cases. $c > 0$ and $c = 0$. In the first case we already proved that $ac < bc$ so, a fortiori, $ac \leq bc$. In the second case, $ac = 0 = bc$. So the conclusion holds in all cases. □

Theorem 1.6. *If $a < b$ and $c < 0$ then $ac > bc$.*

Multiplying by a negative number reverses the inequality. An example is: $2 < 3$ but $-2 > -3$. But an example is not a proof.

Proof. The product of the two positive numbers $b-a$ and $-c$ is positive:

$$-c(b-a) = ac - bc > 0$$

So, $bc < ac$. □

I will talk more about inequalities and sets. We will go over the "triangle inequality" and the "AGM" inequality.

Theorem 1.7 (AGM inequality)**.** *If x, y are nonnegative real numbers then the geometric mean $\sqrt{xy}$ is $\le$ the arithmetic mean $\frac{x+y}{2}$:*

$$\sqrt{xy} \le \frac{x+y}{2}$$

and equality holds only when $x = y$. In general, if x, y are any two real numbers then

$$xy \le \left(\frac{x+y}{2}\right)^2$$

and, again, equality holds only when $x = y$.

The proof is in the book. I want to explain how to use this.

Problem: Show, *without using calculus* that the largest rectangle with perimeter equal to 12 is a 3×3 square.

Answer: A rectangle with width x and height y has area $A = xy$ and perimeter $P = 2x + 2y$. If the perimeter is $P = 12$ then $x + y = 6$. So, the AGM inequality says:

$$A = xy \le \left(\frac{x+y}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 3^2 = 9$$

So, the area is at most 9 and it is equal to 9 only if $x = y = 3$ which means the rectangle is a square.

Definition 1.8. The *absolute value* of a real number x is defined to be:

$$|x| := \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is the distance from x to 0.

There is another equation for absolute value which does both cases at the same time:

$$|x| = \sqrt{x^2}$$

Problem: Prove that $|x||y| = |xy|$.

Theorem 1.9 (Triangle inequality)**.** *If x, y are real numbers then*

$$|x + y| \le |x| + |y|$$

Proof. When you want to prove an equation of inequality with absolute values, you first need to get rid of the absolute value sign somehow. The idea is to square both sides. But this is WRONG!! You cannot square both sides of an equation that you do not know is true. Instead we do a proof by contradiction.

In a standard proof by contradiction, we are saying: Either the statement is true or it is false. To prove the first we assume the second and get a contradiction.

Suppose by contradiction that

$$|x+y| > |x| + |y|$$

Now, we have an assumption. Since both sides are nonnegative we can square them to get:

$$x^2 + 2xy + y^2 > (|x| + |y|)^2 = |x|^2 + 2|x||y| + |y|^2 = x^2 + 2|xy| + y^2$$

Subtracting $x^2 + y^2$ from both sides we get:

$$2xy > 2|xy|$$

which is a contradiction since $|xy| \geq xy$ by definition of absolute value. This contradiction proves the theorem. □

Next, I will do the last unexplained problem from the first day:

Find the set S of all positive real numbers x so that

$$\left|\frac{x-4}{x}\right| \leq 2$$

Hint: first get rid of the absolute value signs.

To eliminate the absolute value sign, we need to know that $|z| \leq 2$ means $-2 \leq z \leq 2$.

For the case at hand we get:

$$-2 \leq \frac{x-4}{x} \leq 2$$

Let's pause to consider the logic of *solving an equation* vs *proving and equation*

(1) To solve an equation, we *assume that it is true*, i.e., that we have a solution x. Then we deduce what x must be.
(2) To prove an equation, we *assume that it is not true*, then we get a contradiction proving that "not true" is not an option.

Multiply by x which is given to be positive, we get

$$-2x \leq x - 4 \leq 2x$$

Subtract x from all three terms to get

$$-3x \leq -4 \leq x$$

This is two statements:

$$-4 \leq x \text{ and } -3x \leq -4$$

We are looking for the set of all positive x satisfying both of these inequalities. The first is true for any positive x. For the second, we reverse the signs to get:

$$3x \geq 4$$

Divide by 3 to get

$$x \geq \frac{4}{3}$$

The logic is the following: We started with the assumption that $x \in S$ and we ended with the conclusion that $x \geq 4/3$. So, what this proves is that S is contained in the set

$$A = [4/3, \infty) = \{x \in \mathbb{R} \mid x \geq 4/3\}$$

To prove that this is the answer, we also need to prove that $A \subseteq S$. Let me explain the set theory and then come back to this.

Definition 1.10. A set S is *contained in* a set T if every element of S is an element of T. Two sets S and T are *equal* if they are contained in each other.

Example: If exactly the same students are registered in Math 1 and Econ 1 then the two class, *as sets of students*, are equal.

Getting back to the problem, to show that $A \subseteq S$ we take an element $x \in A$ (a letter representing an arbitrary element of A) Then we need to show that $x \in S$, i.e. that it satisfies the inequality.

The assumption is $x \in A$ which means $x \geq 4/3$. Multiply both sides by $3/x$ to get

$$3 \geq \frac{4}{x} > 0$$

Subtract 1 from everything:

$$2 \geq \frac{4}{x} - 1 > -1$$

So,

$$\left|\frac{4}{x} - 1\right| \leq 2$$

But

$$\left|\frac{4}{x} - 1\right| = \left|\frac{4}{x} - \frac{x}{x}\right| = \left|\frac{4-x}{x}\right| = \left|\frac{x-4}{x}\right|$$

So, $x \in S$. This proves $A \subseteq S$. Since we already know that $S \subseteq A$ we conclude that $A = S$.

1.3. **Worksheet:** Find the set S of all real numbers x so that

$$\left|\frac{x+4}{x}\right| < 1$$

and prove that your answer is the correct set. (It is understood that $x \neq 0$.)

Hint: First, get rid of the absolute value sign. Then consider separately the two possibilities: $x > 0$ and $x < 0$. The solution has three steps.

Step 1: Find the set of all positive x satisfying the inequality above. Show that there are no positive solutions. This set is empty!

Step 2: Find the set of all negative x satisfying the inequality.

It helps to write $x = -y$ in the second case.

Step 3: Prove that your answer is correct. If you did it correctly, what you have so far is that the solution set S is contained in your answer which I will call A. So, you need to prove that A is contained in S. To do this, take any element $x \in A$ and show that it is in S.

second problem

Prove, by contradiction: *If x, y are real numbers with product $xy = 10$ and $|x| > 2$ then $|y| \leq 5$*

Start with the negation of the conclusion. Then you have three assumptions. What are they?

Next, using the theorem from class arrive at a contradiction.

Finally, you need to write the conclusion, namely that the contradiction proves the result.

If you finish this you can start doing HW1.

1.4. **Homework 1.** HW1 is due next Thursday. The rules for the homework are: You can work in groups but you have to hand in your own version of the answer in your own words. Those words are very important since this is a writing course! Don't try to be wordy but you must write in *complete sentences*. My wife (also a math teacher) says: Write the answer so that your peers can understand it.

(1) Find the set S of all x satisfying the inequality
$$x^2 - 4x + 5 \leq 10$$
Hint: First make the inequality and equality and solve it. Are your solutions to the equality in the set S or not?

(2) Prove that your answer to the first problem is correct. (I haven't explained how to do this yet.)

(3) Prove by contradiction that if $xy \leq 9$ and $x > 3$ then $y < 3$.

(4) Find an example to show that the following statement is NOT true: *If* $xy \geq 9$ *and* $x < 3$ *then* $y > 3$. [Write your answer in words: "Let ... This show that ..."]

(5) Find the set T of all real numbers x so that
$$\left|\frac{x}{x-2}\right| < 4$$
[Hint: Divide into two cases: $x > 2$ and $x < 2$.] Show that T is equal to the set you name as your answer.

(6) You own land along a straight river. You have 400 meters of fence. Prove using AGM (without using calculus!) that the largest rectangular area that you can fence in from three sides using your fencing material (using the river as your fourth side) is 100×200. [Hint: Take a $x \times 2y$ rectangle where $2y$ is the distance along the river.]

1.5. **equality of sets.** (This part I will probably not get to in class. But you need it for your homework. I will explain it on Monday.)

Definition 1.11. A set S is *contained in* a set T if every element of S is an element of T. Two sets S and T are *equal* if they are contained in each other.

Here is an example.

Show that the set S of solutions of the inequality

$$x^2 - 4x < 0$$

is the open interval $S = (0, 4) = \{x \in \mathbb{R} \,|\, 0 < x < 4\}$.

The problem is to show that the two sets S (the solution set) and $T = (0, 4)$ are equal. To do this we show that $S \subseteq T$ and $T \subseteq S$.

(1) First, we show $T \subseteq S$. This mean that the elements of T are in S. So, suppose $x \in T$. Then $x < 4$. So $x - 4 < 0$. Multiplying both sides by $x > 0$ we get

$$x^2 - 4x = x(x - 4) < x(0) = 0$$

So, $x \in S$ which proves $T \subseteq S$.

(2) Next, take any element $x \in S$. This means x is some solution of the inequality above. So:

$$x(x - 4) < 0$$

We need to prove that $x \in (0, 4)$. But our assumption $x(x-4) < 0$ implies that one of the factors is positive and one is negative. There are two possibilities:

(a) $x > 0$ and $x - 4 < 0$ which implies $x < 4$. This is what we want to show.

(b) $x < 0$ and $x - 4 > 0$ which implies $x > 4$ is positive. This is a contradiction.

Since the second case is impossible, the first is true. So $x \in T$. This shows $S \subseteq T$

Since $S \subseteq T$ and $T \subseteq S$ we conclude that $S = T$.

2. More about sets

First you start with a big set U called the "universe" consisting of all of the things your are considering. For example, we could take

$$U = \mathbb{R} = \text{ the set of all real numbers}$$

W can talk about Union, intersection, complement, difference of subsets of U.

(1) *Union* The union of two sets $A \cup B$ is the set of all $x \in U$ so that either $x \in A$ or $x \in B$. Normally student say "and" instead of "or" when talk about the union.
(2) *Intersection* The intersection of two sets $A \cap B$ is the set of all $x \in U$ so that $x \in A$ and $x \in B$.
(3) *Complement* The complement of A is the set of all $x \in U$ so that x is not in A ($x \notin A$). The complement of A is denoted A^c.
(4) *Difference* The difference $A - B$ is the set of all $x \in A$ so that x is not in B.

$$A - B = A \cap B^c$$

Problem: Show that for any two sets A and B we have:

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$$

You can draw a picture, but a picture is not a proof.

Proof. To show that two sets are equal you take an arbitrary element of one set and show that it is in the other set.

(1) Take any $x \in (A \cup B) - (A \cap B)$ then we need to show that x is in the other set.

We are given that x is in $A \cup B$ but not in $A \cap B$. So, x is either in A or B but not not both. Therefore, either x is in A but not in B ($x \in A - B$) or x is in B but not in A ($x \in B - A$). Therefore, x is in the union of these two sets. So

$$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$$

(2) Take any $x \in (A - B) \cup (B - A)$. Then either $x \in A - B$ or $x \in B - A$. In the first case, $x \in A - B$, we get that x is in A and therefore in the union but x is not in the intersection since it is not in B. The secon case is similar. So $x \in A \cup B - A \cap B$. This shows that

$$(A \cup B) - (A \cap B) \supseteq (A - B) \cup (B - A)$$

Since the two sets are contained in each other, they are equal by definition of equality of sets. □

One interesting example of a set is the set of all subsets of a set A. This is called the *power set* of A and denoted $\mathcal{P}(A)$. For example, if $A = \{a, b\}$ then A has 4 subsets: $A, \{a\}, \{b\}, \emptyset$. In words: the entire set, the set containing only a, the set containing only b and the empty set. This means the power set is:

$$\mathcal{P}(A) = \{A, \{a\}, \{b\}, \emptyset\}$$

Theorem 2.1. *If A has n elements then the power set has 2^n elements.*

The reason for this is that every element of the power set can be represented by a string of n 0's and 1's where the 1's indicate which elements are in the set. For example: if $A = \{a, b, c, d, e\}$ then 00101 represents the subset $\{c, e\}$ since there are 1's in the third and 5th entry. This subset is an element of $\mathcal{P}(A)$.

On the other hand, there are exactly 2^n binary sequences of length n. We will prove this by induction later. For now, we should just take it as "obvious." Therefore there are 2^n subsets.

Definition 2.2. The *Cartesian product* of two set $S \times T$ is the set of all ordered pairs (s, t) where $s \in S$ and $t \in T$. For example,

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \,|\, x, y \in \mathbb{R}\}$$

This is the xy-plane.

Notation:

$\mathbb{R}$ is the set of all real number

$\mathbb{Z}$ is the set of all integers

$\mathbb{N}$ is the set of all natural numbers. This not well-defined. "well defined" means there is a precise membership criterion. The question "Is 0 a natural number?" does not have a consistent answer. In our book, 0 is not a natural number. If we agree on that then $\mathbb{N}$ is well-defined.

$\mathbb{Q}$ is the set of rational numbers:

$$\frac{a}{b}$$

where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. (If a, b have no common factors then the fraction is called "reduced")

2.1. **Worksheet.** (a) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Hint: start by letting x denote an element of the first set and show that x is in the second.

(a1) If $x \in A \cap (B \cup C)$ then what do you know about x? Is x in A? in B? in C?

(a2) Going the other way, you start with an element of the second set and show that it is in the first set. There are two cases. What are they?

(b) Let $A = \{a, b, c\}$
(b1) List all the subsets of A.
(b2) Write binary expressions for each of these subsets.

(c1) Draw the set $\mathbb{Z} \times \mathbb{N}$ in the plane.
(c2) What is $\mathbb{Z} \times \mathbb{N} \cap \mathbb{N} \times \mathbb{Z}$?
(c3) If A, B are subsets of $\mathbb{R}$ then prove that

$$(A \times \mathbb{R}) \cap (\mathbb{R} \times B) = A \times B$$

2.2. **functions.**

Definition 2.3. A function or *mapping* f from a set A to a set B is written:

$$f : A \to B$$

This is defined to be a rule which assigns one element of B to every element of A. A is called the *domain* and B is called the *target*.

Example 2.4. Let $A = \{a, b, c\}$ and $B = \mathbb{Z}$ and define the function $f : A \to \mathbb{Z}$ by $f(a) = 5, f(b) = 2, f(c) = 5$. Notice that this function takes the value 5 twice.

If $f(a) = b$ then we say:
b is the *value* of the function f at a and
b is the *image* of a under f.
Instead of naming the target set B we can say "B-valued function"
For example if we say "f is a real-valued function on A" we mean:

$$f : A \to \mathbb{R}$$

In the example above, f is an integer valued function on A.

Definition 2.5. The *image* of a function f is the set of all values that it takes:

$$image(f) = f(A) = \{f(a) \,|\, a \in A\}$$

First a simple example to explain the language. Take the function

$$g : [2, \infty) \to \mathbb{R}$$

given by

$$g(x) = x^2 + 5$$

Here the domain is the half closed interval

$$[2, \infty) = \{x \in \mathbb{R} \,|\, x \geq 2\}$$

The image of this function is the set of all real numbers ≥ 5:

$$g([2, \infty)) = [9, \infty)$$

Problem: *Prove that this is the image (without using calculus)*

Answer: [*with commentary in italics*] To show that these sets are equal, we take an element of each set and show it is in the other.

1) Take any element of the image $g([2, \infty))$. By definition this is $g(x)$ where $x \geq 2$. Since $x \geq 2$ we have $x^2 \geq 4$. Adding 5 to both sides we get $x^2 + 5 \geq 9$. But this is $g(x)$:

$$g(x) = x^2 + 5 \geq 9$$

This proves that the image of g is contained in the set $[9, \infty)$.

2) Now take any $y \in [9, \infty)$. *(You can use any letter. I use y to remind me that this is supposed to be $y = g(x)$.)*

Then we need to show that it is in the image of the function g. *This means we need to solve the equation*

$$y = x^2 + 5$$

and show that our solution is in the domain. The equation is easy to solve:

$$x = \sqrt{y - 5}$$

We just need to show that this x is in the given domain of g.

We are given $y \geq 9$, so $y - 5 \geq 4$ and therefore $\sqrt{y-5} \geq 2$. So $x = \sqrt{y-5}$ is in the domain of g and

$$g(x) = x^2 + 5 = \left(\sqrt{y-5}\right)^2 + 5 = (y - 5) + 5 = y$$

Therefore, y is in the image of g. Since y is an arbitrary element of $[9, \infty)$, this shows that $[9, \infty)$ is contained in the image of g.

Since the image contains this set and this set is contained in the image we conclude that the two sets are equal:

$$image(g) = g([2, \infty)) = [9, \infty)$$

[*"im" is an abbeviations for "imaginary" so don't use "im" for image.*]

Definition 2.6. The *graph* of a function $f : A \to B$ is the subset of the Cartesian product $A \times B$ consisting of all ordered pairs $(a, f(a))$ where $a \in A$.

Definition 2.7. A subset S of $\mathbb{R}$ is called *bounded* if there is a positive real number $M \in \mathbb{R}$ so that $|x| \leq M$ for all $x \in S$. A real valued function on any set $f : A \to \mathbb{R}$ is *bounded* if its image is bounded.

For example, the functions $[0, \infty) \to \mathbb{R}$ given by $f(x) = x^3 - x$, $g(x) = \sqrt{x}$, $h(x) = |x|$ are unbounded but the functions $f(x) = \sin x$ and $g(x) = 1/(1 + x)$ are bounded on the domain $[0, \infty)$.

Problem: Show that $g(x) = 1/(1 + x)$ is bounded on $[0, \infty)$.

Claim: $0 < \frac{1}{1+x} \leq 1$ for all $x \geq 0$.

First, note that this claim proves that the function $g(x)$ is bounded. So, we just need to prove this Claim.

The proof of the Claim is by contradiction. Suppose that $\frac{1}{1+x} > 1$ for some $x \geq 0$. Then we will get a contradiction which will prove the claim. If $\frac{1}{1+x} > 1$ then, multiplying both sides by $1 + x > 0$ we get the inequality $1 > 1 + x$. Subtract 1 from both sides to get $0 > x$ which contradicts the assumption that $x \geq 0$. So, we are done.

2.3. **Worksheet 4.** (a) Let f be the real valued function on the set of positive integers given by the equation

$$f(k) = \frac{k}{k+1}$$

(a1) What are the domain and target of this function?

(a2) Find the image of the function.

(a3) Hopefully, your answer to (a2) is a set. Prove that your set is the image of f.

(a4) Prove that the function f is bounded. What is the bound M? [Use your answer to (a3) to give a short answer to (a4).]

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be the function:

$$g(x) = 1 + 2|x|$$

(b1) Find the image of this function.

(b2) Find the inverse image of the point 5. Find the inverse image of the set $(0, 5)$. [See definition and example on next page.]

Definition 2.8. If $f : A \to B$ is a mapping (=function) and $C \subseteq B$ then the *inverse image* of C is defined to be the set of all $a \in A$ so that $f(a) \in C$:

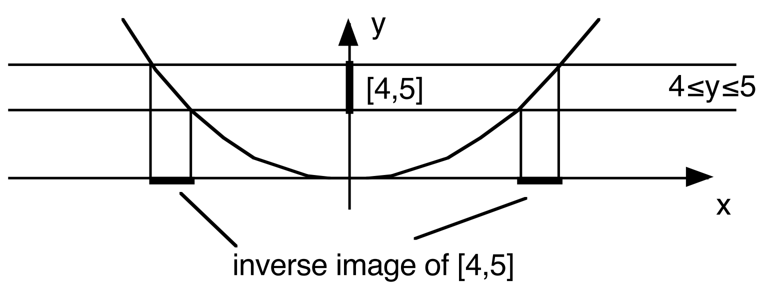
$$f^{-1}(C) = \{a \in A \mid f(a) \in C\}$$

If $b \in B$ is one element then the *inverse image* of b is the set of all $a \in A$ so that $f(a) = b$.

For example, if $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = x^2$, then the inverse image of 4 is the set $f^{-1}(4) = \{2, -2\}$ and the inverse image of the closed interval $[4, 5]$ is the set

$$f^{-1}[4, 5] = [-\sqrt{5}, -2] \cup [2, \sqrt{5}]$$

This can be visualized by figure below. The target set is traditionally drawn on the y-axis. So, the subset $[4, 5]$ is the the segment on the y-axis. You draw horizontal lines at 4 and 5 and the inverse image is the projection to the x-axis of the portion of the graph between these lines. Giving a rigorous proof of concepts given by drawings is very difficult. Normally we write "as illustrated in the figure" or "as suggested by the figure" then we still need to write a proof.



In the book, inverse image is written I_f for example:

$$I_f(4) = \{-2, 2\}$$

However, f^{-1} is standard notation even though it can be confused with inverse function. So, you need to write words: "the inverse image $f^{-1}(C)$ of C"

2.4. **increasing decreasing functions are one-to-one.** Today, I will explain the concept of “one-to-one” or “1-1” or “injective” function using a more familiar concept of increasing and decreasing functions.

Definition 2.9. Suppose $f : S \to T$ is a function where S and T are subsets of the real numbers. We say that f is an *increasing* function if $f(a) < f(b)$ for any $a < b$ in the domain S. We say f is *decreasing* if $f(a) > f(b)$ for all $a < b$.

Here are two examples. Let $S = \mathbb{N}$ (the set of positive integers) and $T = \mathbb{Z}$ (the set of all integers) and $f : \mathbb{N} \to \mathbb{Z}$ is given by

$$f(n) = n^2 + n$$

n	1	2	3	4	5	6	7	...
$f(n)$	2	6	12	20	30	42	56	...

This function is increasing. I will prove it later.

Let $g : \mathbb{Z} \to \mathbb{R}$ be given by

$$g(n) = n^3 - n$$

n	...	-3	-2	-1	0	1	2	3	...
$g(n)$	...	-24	-6	0	0	0	6	24	...

This function is neither increasing nor decreasing. The reason is very simple. The numbers $0, 1$ are in the domain $\mathbb{Z}$ and $0 < 1$ but it is not true that $g(0) < g(1)$ since they are equal:

$$g(0) = 0 = g(1)$$

Another simple example: The function

$$h : \mathbb{R} \to \mathbb{R}$$

given by $h(x) = |x|$ is not increasing or decreasing since

$$h(-1) = 1 = h(1)$$

To prove that a function h is NOT increasing, you need to find two particular numbers (not letters) a, b in the domain so that $a < b$ but $h(a)$ is not less than $h(b)$. In other words, $h(a) \geq h(b)$. So, when I pick the particular numbers $a = 0, b = 1$ for the case of the function g and calculate that $g(0) = g(1)$ then that proves that g is not increasing. It also proves that g is not decreasing since if it were then $g(0)$ would be greater than $g(1)$. (This is the subjunctive case exploring the hypothetical case that g is decreasing.)

It is easy to show that a function is NOT increasing or NOT decreasing since you just need two particular numbers and two calculations. To show that a function IS increasing (or decreasing) you have to use letters representing all possible pairs of numbers in the domain.

Here is the proof that $f(n) = n^2 + n$ is increasing on the domain $\mathbb{N}$.

Proof. Suppose that a, b are any two positive integers and $a < b$. Then we will show that $f(a) < f(b)$. This will prove that f is increasing.

Since $0 < a < b$ we know that $a^2 < b^2$. Using the *addition of inequalities* rule proved below we get

$$a^2 + a < b^2 + b$$

showing that f is increasing on its domain. □

In the future, we can use the *addition of inequalities* rule:

Theorem 2.10. *If $a < b$ and $x < y$ then $a + x < b + y$*

Proof. Adding x to both sides of the first inequality and b to both sides of the second inequality we get:

$$a + x < b + x < b + y$$

□

Definition 2.11. A function $f : A \to B$ is called *injective* or *one-to-one* or 1-1 if for any $a \neq b$ in the domain A, $f(a) \neq f(b)$.

Note: Every function takes one x to one y. An injective function takes two xs to two ys.

Theorem 2.12. *If a function $f : \mathbb{R} \to \mathbb{R}$ is increasing (or decreasing) then it is 1-1.*

Proof. Choose letters representing two different ("distinct") arguments, then show that you get two different values. Let a, b be two distinct real numbers. Then either $a < b$ or $a > b$. We may assume by symmetry that $a < b$. Since f is increasing $f(a) < f(b)$. In particular, $f(a) \neq f(b)$. Therefore, f is 1-1. □

The beauty of this type of mathematics is that we have a very short but completely rigorous proof written in a professional style. (The italicized comment is pedigogical, not part of the proof.)

Problem: Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - 2x$ is not 1-1.

This is easy: Take $0, 2$. These are two distinct elements of the domain but $f(0) = 0 = f(2)$ So the function is not 1-1.

Note that to show that a function is NOT 1-1 you take two particular numbers and show they have the same image. To show that a function is 1-1, you take letters representing two distinct elements of the domain.

2.5. **Worksheet 5.** (a) Don't look at the notes. Write the definition of an increasing function.

(a1) Give a new example of an increasing function (different from the ones that we discussed). [The idea is to find a very simple example. In mathematics we value originality and clarity not complexity.]

(a2) Give a new example of a function which is not increasing. [Here is an example. You need to find a different one. Let $A = [0, 1]$, $B = \mathbb{R}$, $f : [0, 1] \to \mathbb{R}$ is given by $f(x) = 5$ for all x. This is not increasing since $0 < 1$ in the domain but $f(0) = 5 = f(1)$.]

(b) Let $A = [1, 4]$. Let $g : A \to \mathbb{R}$ be the function given by

$$g(x) = 2x^2 + \sqrt{x}$$

Show that g is increasing without using calculus. [Let $1 \leq a < b \leq 4$ and prove that $g(a) < g(b)$.]

What is the inverse image of 0? [Are there any numbers in the interval $[1, 4]$ so that $g(x) = 0$?]

(c) Let $B = [1, 3]$ and $h : B \to \mathbb{R}$ the function given by

$$h(x) = x + \frac{6}{x}$$

Show that h is not decreasing. [Find two specific numbers $a < b$ in the domain and calculate $h(a), h(b)$.]

Is h 1-1? [Does it always take two xs to two ys or are there two xs which give the same y?]

2.6. **Homework 2.** This homework is due next Thursday, Feb 4. Make sure to write complete sentences for all your answers. Write the conclusion for each problem in a sentence at the end of each answer. [For example, "This shows that the function h is increasing."]

(1) Let C be the set of all students in our class. (To make this well-defined, these are the students enrolled at time $T = 1$pm on Jan 28, 2010 in Math 23b at Brandeis) Let U be the set of all undergraduates at Brandeis (more precisely, people who are enrolled as undergraduate students at the university at time T). Let A be the set of all $x \in U$ so that x is a senior at time T. In words, what is $(U - A) \cup C$? Are you a member of this set? What is the complement of this set (in U)?

    What is $(C - A) \cap U$. Are you a member of this set? Can you write these sets with fewer symbols?

(2) Prove that $(A \cup B) \cap (C \cap D) = (A \cap C \cap D) \cup (B \cap C \cap D)$.

(3) Find the power set of the empty set.

(4) Let A be the three point set with elements a, b, c. Let $f : A \to \mathbb{Z}$ be the function given by $f(a) = 1, f(b) = 5, f(c) = 5$.

    Find the graph of f.
    What is the image of f?
    What is the inverse image of 5?
    Is this function 1-1? Prove it.

(5) Let $B = \{a, b\}$ and suppose that $g : B \to \mathbb{R}$ is the function $g(a) = 2, g(b) = 3$. Then it does not make sense to ask if this function is increasing or decreasing. Why not?

(6) Explain why any real valued function on a finite set is bounded. [First, explain what these words mean in symbols: This means $f : A \to B$ where ....]

(7) Find a function $f : \mathbb{R} \to \mathbb{R}$ which is increasing and bounded and prove it.

3. Logic

3.1. Quantifiers. There are two types of quantifiers: The universal quantifier $\forall$ and the existential quantifier $\exists$.

3.1.1. *universal quantifier.* $\forall x \in S$ says that the property P which follows holds for all x in the set S.

For all integers n, the number $n(n+1)$ is even.

In symbols:

$$(\forall n \in \mathbb{Z}) n(n+1) \in 2\mathbb{Z}$$

where $2\mathbb{Z}$ is the set of even integers. Here the property is $P(n) =$ "$n(n+1)$ is even." The general syntax is:

$$\boxed{(\forall x \in S) P(x)}$$

We can also make precise statements which are false. For example: "The square of every real number is negative" is, in logical notation,

$$(\forall x \in \mathbb{R}) x^2 < 0$$

Although the statement is false, this is a correct translation of the English sentence into symbols.

3.1.2. *existential quantifier.* $\exists x \in S$ says that there is at least one x in the set S which satisfies the property that follows. In symbols it is:

$$\boxed{(\exists x \in S) P(x)}$$

In words you would write: There is an x in the set S so that property $P(x)$ holds. In the example above, if we change $\forall$ to $\exists$ then we get:

$$(\exists n \in \mathbb{Z}) n(n+1) \in 2\mathbb{Z}$$

In words: There is an integer n for which $n(n+1)$ is even.

Warning! It seems obvious that, if a statement is true for all x then it is true for at least one x. But the existence statement tells you something else: It states that the set S is not empty.

For example: "Every floating city has a castle" does not imply that "There is a floating city which has a castle"

The first "for all" statement makes more sense if the sentence is in the subjunctive case: "If there were a floating city then it would have a castle". Generally, the subjunctive case is appropriate in "for all" statements when existence is in question.

3.1.3. *order of quantifiers.* "If $x > 0$ is real then $x = y^2$ for some $y \in \mathbb{R}$" The property is

$$P(x, y) : \quad x = y^2$$

This becomes a statement when both variables are quantified:

$$(\forall x \in \mathbb{R}_{>0})(\exists y \in \mathbb{R})x = y^2$$

Very often the sentence, at least in English, has quantifiers in the wrong order, at least from a logical point of view. Please let me know if this is true in other languages. For example:

"There is a quiz for every student."

In logical order this is:

$$(\forall x \in S)(\exists c \in C) \text{ } c \text{ is for student } x$$

Notice that sentence has only quantifiers (in the wrong order) and the logical statement has a statement about c and x. Here S is the set of students in the class and C is the set of copies of the quiz.

In the other logical order, the meaning is different:

$$(\exists c \in C)(\forall x \in S) \text{ } c \text{ is for student } x$$

This says "There is one copy of the quiz for all the students."

3.2. **Worksheet.** Translate these symbols into words and determine if the statement is true or not.

(1) $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})x^n > 0$
(2) $(\exists n \in \mathbb{Z})(\exists x \in \mathbb{R})x^n = 2$

Write these statements in the form (quatifiers)statement. Then write, in words, what the sets are. ("where S is the set of all ...")

(1) For every real number x you can find an integer n which is greater than x^2.
(2) I have an answer for each of your questions.
(3) I have one answer for all of your questions.

0. Quiz 0

Rules: Closed book. Notes on one sheet of letter sized paper allowed. (Prepare one for the next quiz!)

(1) What is the statement of the triangle inequality? Give an example.

The triangle inequality says: For all real numbers x, y we have

$$|x + y| \leq |x| + |y|$$

for example, if $x = 5$ and $y = -2$ then $|x + y| = |5 - 2| = 3$ is $\leq |x| + |y| = 2 + 3 = 5$.

(2) Find the solution set of the inequality $x^2 + 4x + 4 \leq 9$. (No proof needed.)

The solution set is the closed interval $[-5, 1]$.

The explanation (which was not required) is that $(x + 2)^2 \leq 9$ gives $|x + 2| \leq 3$ which is the same as $-3 \leq x + 2 \leq 3$. Subtract 2 from all three terms to get the answer. (This explanation proves $S \subseteq [-5, 1]$.)

(3a) What is the definition of a bounded function?

(3b) What is the definition of an increasing function?

(3c) Find an example of an increasing function $f : \mathbb{R} \to \mathbb{R}$ which is bounded.

A *bounded function* is a function $f : A \to \mathbb{R}$ with the property that there is a positive real number M so that $|f(a)| \leq M$ for all $a \in A$. An *increasing function* is function $f : B \to \mathbb{R}$ whose domain is a subset $B \subset \mathbb{R}$ so that $f(a) < f(b)$ whenever a, b are two elements of B so that $a < b$.

An example of an increasing bounded function if $f(x) = x$ on the domain $B = [0, 1]$. This is bounded by $M = 1$ since $|x| \leq 1$ for all x in the interval $[0, 1]$. The function is increasing since $a < b$ implies $f(a) = a < b = f(b)$.

(4) Prove, by contradiction, that if $A - B = A$ then $A \cap B$ is the empty set.

Suppose by contradiction that $A \cap B$ is nonempty. Then it has an element, say $a \in A \cap B$. Then $a \in A$ and $a \in B$ which implies that $a \notin A - B$ since the set $A - B$ is the set of all $x \in A$ so that $x \notin B$. Therefore, A is not contained in the set $A - B$ which contradicts the assumption that $A = A - B$. This proves that $A \cap B$ cannot be nonempty. So, it must be empty as claimed.

3.3. **symbolic logic.**

Definition 3.1. A **statement** is an expression with a well-defined truth value: True or False.

This means a statement is true (T) or false (F). "Well-defined" means it is not ambiguous.

If p is one statement, it is either true or false. If q is another statement then there are 4 possibilities:

p	q
T	T
T	F
F	T
F	F

Sometimes, not all of these are actually possible. For example if p is the statement $x > 0$ and q is the statement $x^2 > 0$ then the second case is not possible

$x > 0$	$x^2 > 0$	*example*
T	T	$x = 1$
T	F	not possible
F	T	$x = -1$
F	F	$x = 0$

This represents the true statement

$$(\forall x \in \mathbb{R})(x > 0) \Rightarrow (x^2 > 0)$$

However, there is more that I need to explain about this.

3.3.1. *logical connectives.* We will use five operations: $\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow$ The meaning of these symbols is defined by the truth tables which I will draw.

$\neg p$ means "not p".
If p is true then $\neg p$ is false.
If p is false then $\neg p$ is true.

p	$\neg p$
T	F
F	T

$p \wedge q$ means "p and q" which means they are both true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p \vee q$ means "p or q" which means one or the other or both are both true. In math "or" is inclusive. So, the top line in the truth tables says: If p is true and q is also true then $p \vee q$ is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

There is only one case where $p \vee q$ is false. So, it is probably going to be easier to prove by contradiction since there is only one case to eliminate. To prove $p \wedge q$ by contradiction, we need to rule out three other possibilities.

$p \Rightarrow q$ means "p implies q". We discussed this in length because this in not the same in English as it is in the language of Math. First the truth table:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In mathematics we adopt the concept of **vacuous truth**. This means that a statement is true when there are no cases of it. The statment $p \Rightarrow q$ is interpreted as being completely hypothetical. It says "If p were true then q would be true." It doesn't say anything about what happens when p is false. So, when p is False, the statement $p \Rightarrow q$ is true. This is by definition. It does not necessarily make any common sense.

The precise meaning of the logical symbol $p \Rightarrow q$ is given by the truth table. The only way it can be false is if p is true and q is false.

So, $p \Rightarrow q$ is equivalent to $\neg p \vee q$. The following truth table proves this.

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

The example I gave in class was the famous quote from the movie "The Godfather" (This occurs at the beginning of the movie.) There is a conversation about how the Godfather coerced someone to sign a contract by telling him "Either his signature or his brains would be on the contract." This was a colorful way to say that if he did not sign the document then he would be shot. Here:

p is "you don't sign the contract"
$\neg p$ is "you sign the contract"
q is "you will be shot"

What the Godfather said was $\neg p \vee q$. What he meant was $p \Rightarrow q$ which is locally equivalent.

Logical equivalence is $p \Leftrightarrow q$. This is also the same as $(p \Rightarrow q) \wedge (q \Rightarrow p)$

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Here is the truth table showing that $p \Leftrightarrow q$ is the same as $(p \Rightarrow q) \wedge (q \Rightarrow p)$

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

3.4. **worksheet.** You need to (1) understand the meaning of symbolic logical statements and (2) be able to translate them into English and (3) Translate statements into symbolic notation.

(1a) Write the truth table for the expression $p \vee (q \wedge r)$. (Use this in (2).)

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

(1b) Write the truth table for the expression $(p \Rightarrow q) \Rightarrow \neg r$.

p	q	r	$p \Rightarrow q$	$\neg r$	$(p \Rightarrow q) \Rightarrow \neg r$
T	T	T	T	F	**F**
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	**F**
F	T	F	T	T	T
F	F	T	T	F	**F**
F	F	F	T	T	T

(2) Suppose that p is the statement $x > 1$ and q is the statement $y > 2$ and r is the statement $0 < x - y < 3$. Write the following in sentences without $p, q, r, \forall, \exists$ and determine if they are true statements.

(2a) $(\exists x, y \in \mathbb{R}) p \vee (q \wedge r)$

(2b) $(\forall x, y \in \mathbb{R})(p \Rightarrow q) \Rightarrow \neg r$

Here the statement "If $x > 1$ implies $y > 2$ then it is not true that $0 < x - y < 3$" just doesn't make any sense. You need to replace it with the logically equivalent statement: "If $x \leq 1$ or $y > 2$ then it follows that $x - y$ is not in the open interval $(0, 3)$" This wording makes sense and we can see that the statement is false when universally quantified.

(3) Write in symbolic notation (It does not matter if it is true or not): On any day that there is a snow storm tomorrow and classes are cancelled, I will stay in my room and study math! ($p =$ "There is a snow storm on day x") The hint should have said on day $x + 1$ since it says "tomorrow"

3.5. **negation and proof.**

3.5.1. *equations with variables.* First, we should go to a more accurate notation. When a statement has a variable, such as "$x \geq 2$" then it is sometimes true and sometimes false. So, we write $P(x)$ for short and say: $P(x)$ is NOT a statement.

Why? Because a statement is either true or false. Not sometimes one and sometimes the other. When x is quantified then this becomes a statement:

$$(\forall x \in \mathbb{R})P(x)$$

$$(\exists x \in \mathbb{R})P(x)$$

The first statement is false. The second is true.

When you do a proof by contradiction then you assume that the thing you are trying to prove is false and get a contradiction. So, we need to know what it means when a statement is false.

3.5.2. *negation of quantifiers.* *The negation of* $(\forall x \in S)P(x)$ *is* $(\exists x \in S)\neg P(x)$.

The original statement is that $P(x)$ is true for all $x \in S$. "For all" means no exceptions. The negation of this statement is that there is an exception. Exception means $\neg P(x)$. $(\exists x \in S)\neg P(x)$ means there is an exception to the equation $P(x)$ which is in the set S.

For example, it is not true that $(\forall x \in \mathbb{N})x \geq 2$ because there is one exception: $x = 1$ is in the set $\mathbb{N}$. So, there is at least one number x in the set $\mathbb{N}$ so that $x \geq 2$ is not true:

$$(\exists x \in \mathbb{N})x < 2$$

Here I substituted $x < 2$ for $\neg(x \geq 2)$.

The negation of $(\exists x \in S)P(x)$ *is* $(\forall x \in S)\neg P(x)$

Here the reasoning is that, if there are no element x of the set S which satisfy the equation $P(x)$ then that is the same as saying that for every x in the set S, $P(x)$ is not true.

3.5.3. *negation of logical connectives.* The rules for negating $p \wedge q$, $p \vee q$, $p \Rightarrow q$, $p \Leftrightarrow q$ are easy from the truth table:

$\neg(p \wedge q)$ is the same as $\neg p \vee \neg q$ and the same is true if there are several statements.

$\neg(p \wedge q \wedge r \wedge s)$ means p, q, r, s cannot all be true. One of them must be false: $\neg p \vee \neg q \vee \neg r \vee \neg s$.

Example: We know that $(\forall x \in \mathbb{R}) x > 0 \vee x < 0 \vee x = 0$. The negation of this statement is

$$(\exists x \in \mathbb{R})(x > 0 \wedge x < 0 \wedge x = 0)$$

This is definitely false.

$\neg(p \vee q)$ is the same as $\neg p \wedge \neq q$

$\neg(p \Rightarrow q)$ is the same as $p \wedge \neg q$.

This is the really important point. The only way that $p \Rightarrow q$ can be false is if p is true and q is false.

Here is an example: Problem: prove by contradiction that for all integers n, $n > 3$ implies $2^2 = 4$.

Proof. In symbols the statement is:

$$(\forall n \in \mathbb{Z})(n > 3) \Rightarrow (2^2 = 4)$$

Suppose by contradiction that this is false. Since the negation of $\forall$ is $\exists$ and the negation of $p \Rightarrow q$ is $p \wedge \neg q$, we have:

$$(\exists n \in \mathbb{Z})(n > 3) \wedge (2^2 \neq 4)$$

This says you can find a number n so that $n > 3$ AND $2^2 \neq 4$. This contradicts the fact that $2^2 = 4$. This contradiction proves that the original statement is true: For all integers n, the equation $n > 3$ implies that $2^2 = 4$. □

$\neg(p \Leftrightarrow q)$ is the same as $p \Leftrightarrow \neg q$

3.5.4. *logic and sets.*

$$(\forall x \in A) x \in B$$

means A is a subset of B.

$$A \cup B := \{x \in U \mid (x \in A) \vee (x \in B)\}$$

So, the statement $C \subseteq A \cup B$ is in logical symbols:

$$(\forall x \in C)(x \in A) \vee (x \in B)$$

We can go backwards to convert equations into sets. For example the true statement:

$$(\forall x \in \mathbb{R}) x \geq 0 \vee x \leq 0$$

In this statement $C = \mathbb{R}$, $A = [0, \infty)$, $B = (-\infty, 0]$. So the statement is equivalent to the set theory equation:

$$\mathbb{R} \subseteq [0, \infty) \cup (-\infty, 0]$$

which is obviously true.

Problem: Convert to logic: $[-5, 4] \cap [1, 5] = [1, 3] \cup [2, 4]$

3.6. **worksheet.** (1) Negations: Write down the negations of these statement. Which is true the statement or its negation?

(1a) $(\forall x \in \mathbb{R})(x \leq 0) \vee (x^2 > 2) \vee (x^3 > 3)$

(1b) $(\exists n, m \in \mathbb{Z})(n \geq 5 \Rightarrow m \leq 4) \wedge n + m \leq 9$

(Note that the negation of $m \leq 4$ is $m \geq 5$ since m is an integer.)

(2) Sets and symbolic logic: Using the expressions $P(x) : x \in A$, $Q(x) : x \in B$, $R(x) : x \in C$, $S(x) : x \in D$ to make the answer to (2a) look like question (2b) and the answer to (2b) look like question (2a).

(2a) Write in symbolic logic: $A \subseteq B \cap (C \cup D)$

(2b) What does the following say in terms of sets contained in other sets?

$$(\forall x \in U)(P(x) \wedge Q(x)) \Rightarrow (R(x) \vee S(x))$$

(2c) What does the following say in terms of sets contained in other sets?

$$(\forall x, y \in U)(P(x) \wedge Q(y)) \Rightarrow (R(x) \vee S(y))$$

Assume given that all four sets A, B, C, D are nonempty subsets of a universe U.

3.7. **answers to worksheet.** (1) Negations: Write down the negations of these statement. Which is true the statement or its negation?

(1a) $(\forall x \in \mathbb{R})(x \leq 0) \vee (x^2 > 2) \vee (x^3 > 3)$

$$(\exists x \in \mathbb{R})(x > 0) \wedge (x^2 \leq 2) \wedge (x^3 \leq 3)$$

$x = 1$ fits this description. So, the negation is true.

(1b) $(\exists n, m \in \mathbb{Z})(n \geq 5 \Rightarrow m \leq 4) \wedge n + m \geq 10$

(Note that the negation of $m \leq 4$ is $m \geq 5$ since m is an integer.) (oops! this is not the same question!)

$$(\forall n, m \in \mathbb{Z})(n \geq 5 \wedge m \geq 5) \vee n + m < 10$$

This negation is not true (making the original statement true)since the numbers $n = 4, m = 10$ do not satisfy either of the condition: $(n \geq 5 \wedge m \geq 5)$ is not true and $n + m < 10$ is not true.

(2) Sets and symbolic logic: Using the expressions $P(x) : x \in A$, $Q(x) : x \in B$, $R(x) : x \in C$, $S(x) : x \in D$ to make the answer to (2a) look like question (2b) and the answer to (2b) look like question (2a).

(2a) Write in symbolic logic: $A \subseteq B \cap (C \cup D)$

$$(\forall x \in A)Q(x) \wedge (R(x) \vee S(x))$$

(2b) What does the following say in terms of sets contained in other sets?

$$(\forall x \in U)(P(x) \wedge Q(x)) \Rightarrow (R(x) \vee S(x))$$

$$(A \cap B) \subseteq (C \cup D)$$

(2c) What does the following say in terms of sets contained in other sets?

$$(\forall x, y \in U)(P(x) \wedge Q(y)) \Rightarrow (R(x) \vee S(y))$$

Assume given that all four sets A, B, C, D are nonempty subsets of a universe U.

This says that $A \subseteq C$ or $B \subseteq D$.

3.8. **Homework 3.** Due Thursday, Feb 11 (also on LATTE).

HW3 is Chap 2 problem numbers 2,3,19,21,25,47,48,49,52

On the last problem 2.52: Convert the statement into logic then prove it.

There is one bonus question:

Bonus question: Write in symbolic language the outcome of one move in the following game: (The object of this game is to get rid of all your cards.) If you play a 2 your opponent picks up two cards unless he can immediately play another 2 in which case you pick up 4 card unless you have a 2 in which case your opponent picks up 8 cards unless he has the last 2 in which case you pick up 16 card!

3.9. **Review.**

symbolic	*meaning*	*true example*	*false example*
$\forall x \in S$	for all x in S	$(\forall x \in \emptyset)0 = 1^*$	$(\forall n \in \mathbb{Z})\sqrt{n} \in \mathbb{Z}$
$\exists x \in S$	there is (at least one) x in S	$(\exists x \in \mathbb{R})x^2 = 2$	$(\exists n \in \mathbb{N})5 = 7$
$\neg p$	p is not true	$\neg(1 = 2)$	$\neg(\exists x \in \mathbb{R})x^2 < 1$
$p \wedge q$	p and q are both true	$(\forall x \in \mathbb{R})x = x \wedge \lvert x\rvert \geq x$	$(\exists x \in \mathbb{R})x < 0 \wedge x > 0$
$p \vee q$	p or q (or both) are true	$(\forall x \in \mathbb{R})x > 0 \vee x \leq 0$	$(\exists n \in \mathbb{N})n < 0 \vee n^2 = 2$
$p \Rightarrow q$	If p is true then q is true	$(1 = 0) \Rightarrow$ your Mom is a pig*	$(\forall x \in \mathbb{R})x > 2 \Rightarrow x^2 < 9$
$p \Leftrightarrow q$	p and q are either both true or both false	$(\forall x \in \mathbb{R})x = 0 \Leftrightarrow x^2 = 0$	$(\exists n \in \mathbb{Z})n < 3 \Leftrightarrow n^2 > 5$

* These are examples of *vacuous truth*

symbolic	*false example*	*negation*	*negation of example*
$(\forall x \in S)P(x)$	$(\forall n \in \mathbb{Z})\sqrt{n} \in \mathbb{Z}$	$(\exists x \in S)\neg P(x)$	$(\exists n \in \mathbb{Z})\sqrt{n} \notin \mathbb{Z}$
$(\exists x \in S)P(x)$	$(\exists n \in \mathbb{N})5 = 7$	$(\forall x \in S)\neg P(x)$	$(\forall n \in \mathbb{N})5 \neq 7$
$\neg p$	$\neg(\exists x \in \mathbb{R})x^2 < 1$	p	$(\exists x \in \mathbb{R})x^2 < 1$
$p \wedge q$	$(\exists x \in \mathbb{R})x < 0 \wedge x > 0$	$\neg p \vee \neg q$	$(\forall x \in \mathbb{R})x \geq 0 \vee x \leq 0$
$p \vee q$	$(\exists n \in \mathbb{N})n < 0 \vee n^2 = 2$	$\neg p \wedge \neg q$	$(\forall n \in \mathbb{N})n \geq 0 \wedge n^2 \neq 2$
$p \Rightarrow q$	$(\forall x \in \mathbb{R})x > 2 \Rightarrow x^2 < 9$	$p \wedge \neg q$	$(\exists x \in \mathbb{R})x > 2 \wedge x^2 \geq 9$
$p \Leftrightarrow q$	$(\exists n \in \mathbb{Z})n < 3 \Leftrightarrow n^2 > 5$	$p \Leftrightarrow \neg q$	$(\forall n \in \mathbb{Z})n < 3 \Leftrightarrow n^2 \leq 5$

$$A \cup B = \{x \in U \mid x \in A \vee x \in B\}$$
$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$$
$$A^c = \{x \in U \mid \neg(x \in A)\}$$
$$A \subseteq B \iff (\forall x \in A)x \in B \iff (\forall x \in U)x \in A \Rightarrow x \in B$$
$$A = B \iff (\forall x \in U)x \in A \Leftrightarrow x \in B$$

3.10. **tautologies.** A **tautology** is a statement which is always true because of logic. For example: $p \vee \neg p$, $[p \wedge (p \Rightarrow q)] \Rightarrow q$, $\neg q \vee (p \Rightarrow q)$ Another example:

$$(\forall x \in \mathbb{R})x > 0 \vee x \not> 0$$

is a tautology. But

$$(\forall x \in \mathbb{R})x > 0 \vee x \leq 0$$

is not a tautology, it is a property of real numbers.

3.11. **worksheet.** (1) Show that the following are tautologies.

(1a) For any statements p and q either p is true or p implies q.

(1b) $[p \wedge (p \Rightarrow q)] \Rightarrow q$

[There are two ways to show that a statement is a tautology: (a) by using a truth table and showing that the composite statement is true in all cases (b) by logic (using words) which amounts to the same thing but makes more sense. "We know that p is either true or false. If p is true then [the statement is true because ...]. If p is false [the statement is true because ...]. So the statement is true regardless of the truth or falsity of p making it a tautology."

A wordy proof: To show that $p \vee (p \Rightarrow q)$ is a tautology, we will show it hold regardless of the truth values of p and q. The statement p is either True or False. If p is true then the first part of the disjunction $p \vee (p \Rightarrow q)$ is true. If p is false then $p \Rightarrow q$ is true making the second part of the disjunction true. Therefore, either p of $p \Rightarrow q$ is true in all cases making this a tautology.

(2) Induction. Show by induction that positive integers are less than or equal to their squares. [This is simple problem made difficult in order to practice induction.]

(i) Write down the *Induction hypothesis* $P(n)$. For which values of n do you think it is true?

(ii) State the *basis for induction*. This is the first case. Insert the smallest number n into the hypothesis.

(iii) The *induction step* is $P(n) \Rightarrow P(n+1)$. Write down what is the statement $P(n+1)$ which you are going to prove.

(iv) Finally, prove the induction step. (Prove $P(n+1)$ assuming $P(n)$.)

The induction hypothesis is $P(n) : n \leq n^2$ and $n \geq 1$. (Note: n is always and **integer** in induction, not a real number.)

The basis for induction is $P(1)$ which says $1 \leq 1^2$. This is true. The basis case holds.

In the induction step, we assume $P(n)$: $n \leq n^2$. We also assume $n \geq 1$. Then we need to show $P(n+1)$ which says $n+1 \leq (n+1)^2 = n^2 + 2n + 1$. [Be careful! You don't start with this, end with it.]:

We assume by induction that $n \leq n^2$ and $n \geq 1$. Then $2n \geq 2$ and $2n + 1 \geq 3 > 1$. Adding this to $n^2 \geq n$ we get

$$n^2 + 2n + 1 \geq n + 1$$

which is the statement $P(n+1)$. So, the induction is complete and we can conclude that the statement holds for all positive integers n.

4. Mathematical Induction

You start with a statement or formula that you are trying to prove for all positive integers n.

$$(\forall n \in \mathbb{N})P(n)$$

This means we have a sequence of statements:

$$P(1), P(2), P(3), \cdots$$

It is important that this has a starting point.

4.1. **simple induction.**

The induction process called *simple induction* goes as follows.

(1) Show $P(1)$ is true.
(2) Show $P(n) \Rightarrow P(n+1)$ for $n \geq 1$.
$\Rightarrow$ Then $P(n)$ holds for all $n \geq 1$

This says the first statement in the sequence is true and each statement implies the next statement. So, like dominoes, they all fall down and they are all true.

Let's do the example: $P(n) : n < 2^n$ (I did $n \leq 2^n$ is class.)

The equation that we are trying to prove is called the **induction hypothesis (IH)** and is written $P(n)$. In the example, the induction hypothesis is $n < 2^n$.

$$\boxed{\text{induction hypothesis is } P(n) : \quad n < 2^n}$$

Theorem 4.1. *$n < 2^n$ for all positive integers n.*

Induction starts with the **basis** for induction which is the first case that is supposed to be true. This is $n = 1$, given in the theorem. But sometime you need to figure it out yourself.

To show the basis for the induction you need to insert 1 (or whatever is the first value of n) into the induction hypothesis:

$$\boxed{\text{basis is } P(1) : \quad 1 < 2^1 = 2}$$

Since $1 < 2$, the statement $P(1)$ is true.

Next comes the *induction step*. This will explain why the equation we are trying to prove becomes a hypothesis. The induction step is to show:

$$\boxed{\text{induction step: } P(n) \Rightarrow P(n+1)}$$

In this step we *assume* that $P(n)$ holds and then, after some algebraic manipulations, we should conclude that $P(n+1)$ holds. To do this, you need to determine: What is $P(n+1)$?

$$\boxed{\text{wts: } P(n+1) : n+1 < 2^{n+1}}$$

We assume that $P(n)$ holds and that $n \geq 1$. So:

$$n < 2^n$$

And $1 \leq n < 2^n$. So, $1 < 2^n$. Adding the left sides and the right sides, we get:

$$n + 1 < 2^n + 2^n = 2(2^n) = 2^{n+1}$$

So, $P(n+1)$ holds. By induction we conclude that $P(n)$ holds for all $n \geq 1$.

Now, let's write the proof from the beginning using the corrects words:

Proof. We will show that $n < 2^n$ for all positive integers n. First we settle the basis case $n = 1$. In this case the statement becomes:

$$1 < 2^1 = 2$$

which is true. So, the basis case holds. Now, we assume by induction that $n < 2^n$ and $n \geq 1$. Then we need to show that $n + 1 < 2^{n+1}$. To do this note first that

$$1 \leq n < 2^n$$

by the induction hypothesis. Adding this to $n < 2^n$ we get:

$$n + 1 < 2^n + 2^n = 2(2^n) = 2^{n+1}$$

which is what we are trying to prove. This completes the induction and we conclude that the statement $n < 2^n$ holds for all $n \geq 1$. □

4.2. **worksheet.** (1) Strong induction. You have a scale and two piles of weights: 3oz weights and 5oz weights. (Note 16oz= one pound) This problem is similar to the quarter and dime problem but more challenging.

a) First, find integers a, b so that $3a + 5b = 1$. $[1 = 3(2) + 5(-1)]$

b) Show that, if you are allowed to place weights on both sides of the scale then you can measure any integer number of ounces.

If you want to weight n ounces you put the item that you are weighing together with n 5 oz weights on one side of the scale and $2n$ 3 oz weights on the other side.

c) If you can only put weights on one side then show that you can collect n ounces on one side for all $n \geq 8$ by strong induction.

(c1): You need 3 base cases. State them and prove them.

The mathematical statement is, for any integer $n \geq 8$ there exist nonnegative integers a, b so that

$$n = 3a + 5b$$

This is true when $n = 8, 9, 10$ since

$$8 = 3(1) + 5(1)$$
$$9 = 3(3) + 5(0)$$
$$10 = 3(0) + 5(2)$$

The statement in words is: You can weigh the quantities 8,9 and 10 ounces using weights of size 3oz and 5oz on the other side of the scale.

(c2): In standard strong induction you prove $P(n)$ assuming that $P(k)$ holds for all $a \leq k < n$ where $P(a)$ is the base case. For this problem you need 3 base cases.

What is the induction hypothesis for your strong induction in this problem?

1) All integers from 8 to $n - 1$, (including $n - 1$), can be written as a sum of 3's and 5's.

2) We also assume that $n \geq 11$.

(c3) Do the induction step. List all the assumptions. Why are they justified?

The two assumptions are given above. They are justified because we are assuming that we have done all the cases up to $n - 1$. The number n is the next case. Since we did $n = 8, 9, 10$ the next cases are 11 and above.

The induction step is: Since $n \geq 11$, $n - 3 \geq 8$ and we have already done that case. So, you add one 3oz weight to the collection of weights that worked for $n - 3$. This proves the induction and therefore shows that the statement is true for all $n \geq 8$.

4.3. **summation problems.** A typical simple induction problem is to calculate a sum of n terms. For example

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

We use **summation notation** to write this:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

This is the statement. The first case is $n = 0$. This is where there are no summands on the left side. So we have zero on the left. On the right side we also have zero: $0(1)/2 = 0$. So, the base case holds.

Suppose by induction that the equation holds for n. Then we will prove it for $n + 1$. This means we have to prove that

$$1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$$

By the induction hypothesis we have

$$(1+2+\cdots+n)+(n+1) = \frac{n(n+1)}{2}+n+1 = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+2)(n+1)}{2}$$

which is what we needed to prove. Therefore, the summation equation holds for all $n \geq 0$.

Here is another similar problem. Show that the sum of the first n squares is equal to $n(n+1)(2n+1)/6$. Write the equation in summation form.

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Again the basis for the induction is the case $n = 0$. There is zero on the left and zero on the right. So the basis case holds. Next, assume by induction that the equation holds for n and that $n \geq 0$. Then we need to prove it for $n + 1$. The equation we need to prove is:

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

But, by the induction hypothesis we have:

$$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$\frac{n+1}{6}\left[n(2n+1) + 6(n+1)\right] = \frac{n+1}{6}\left[n^2 + 7n + 6\right] = \frac{n+1}{6}\left[(n+2)(2n+3)\right]$$

which is what we needed to prove.

4.4. **strong induction.** In simple induction you assume $P(n)$ and prove $P(n+1)$.

In strong induction you prove $P(n)$ assuming that $n > a$ (where a is the value of n in the basis case $P(a)$) and assuming that $P(k)$ holds for all $k < n$. The idea is that this domino is very heavy, but all the previous dominoes will be putting their weight on it (see figure).

Example Quarters and dimes. Using quarters (25 cents) and dimes (10 cents) you cannot make 5 cents or 15 cents but you can make any other positive multiple of 5 cents. What is the statement in mathematical terms and how do you prove it. You need strong induction with two base cases.

The mathematical statement is that for any integer $n \geq 4$ there exist nonnegative integers a, b so that

$$5n = 10a + 25b$$

The interpretation is: You can make any multiple of 5 cents which is at least 20 cents using dimes and quarters. (a dimes and b quarters.)

There are two base cases: $n = 4$ and $n = 5$. (You need to knock over the first two dominoes by hand. Then the others fall down because they have the weight of at least two dominoes bearing down on them.)

Base case $n = 4$: You can make $4 \times 5 = 20$ cents with two dimes:

$$5(4) = 10(2) + 25(0)$$

And you can make 25 cents with one quarter:

$$5(5) = 10(0) + 25(1)$$

Next, we need to prove $P(n)$ assuming that $n \geq 6$ (since we just did $n = 4$ and $n = 5$) and assuming that $P(k)$ holds for all k so that $4 \leq k \leq n-1$. (This is the list of all of the cases that were done before we came to $P(n)$.) Since $n \geq 6$, $n - 2 \geq 4$. Therefore, by the strong induction hypothesis, $P(n-2)$ holds. So, we can make $5(n-2)$ cents. Add one dime and we have $5n$ cents proving $P(n)$. So, this proves the statement for all $n \geq 4$. The mathematical statement is: By strong induction there are nonnegative integers a, b so that

$$5(n-2) = 10a + 25b$$

Thus

$$5n = 10(a+1) + 25b$$

So, $5n$ is a nonnegative integer linear combination of 10 and 25.

(The noun clause "linear combination" of 10 and 25 mean $10x + 25y$. The "nonnegative integer" adjective clause means that x, y are nonnegative integers.)

4.5. **theory of induction.** The theory behind strong induction is easy to explain. It is based on the following property of positive integers.

Axiom 4.2. *Every nonempty set of positive integers S contains a least element.*

In other words: *If there is a positive integer n satisfying property $P(n)$ then there exists a smallest positive integer n satisfying $P(n)$.*

We call this the **well-ordered** property of the set $\mathbb{N}$. The real numbers don't have this property since, e.g., there is no smallest real number. The set of nonnegative integers is also well-ordered.

Here is an example of the use of this property which also explains the logic behind strong induction.

Theorem 4.3. *There is no rational number n/m whose square is* 2

In other words there do not exist positive integers n, m so that

$$\frac{n^2}{m^2} = 2$$

The book says "$\sqrt{2}$ is not a rational number" which states that there exists a real number whose square is 2 and that that real number is not a rational number.

Proof. Suppose by contradiction that $2 = n^2/m^2$ for some positive integers n and m. This gives:

$$n^2 = 2m^2$$

$P(n)$: n^2 is twice the square of another number.

By the well-ordered property of $\mathbb{N}$, there is a smallest positive integer n with this property. (We say n is *minimal* with this property.)

Since n, m are positive integers, this equation implies two things:

(1) $n > m$ (since $n^2 = 2m^2 = m^2 + m^2 > m^2$)
(2) n is even (if n were odd, n^2 would be odd)

Therefore, $n = 2k$ for some positive integer k. But then

$$n^2 = 4k^2 = 2m^2 \Rightarrow 2k^2 = m^2.$$

So, m^2 is twice the square of another number. So, $P(m)$ is true. But $m < n$. This contradicts the minimality of n. Therefore, there is no positive integer n with this property which proves the theorem. □

This was a strong induction argument in disguise. When we say n is the smallest positive integer with property $P(n)$ we are saying that $P(n)$ does not hold for all positive integers less than n. So, we are proving $\neg P(n)$ by strong induction. And there was *no base case.*

Theorem 4.4. *A finite set A with n elements has exactly 2^n subsets.*

In this example, I will do the proof twice, once usiong strong induction and the second time using the well-ordered property.

Proof. First take the case $n = 0$. So, $A = \emptyset$. The empty set has no proper subset. So $\emptyset$ is the only subset of A and $\mathcal{P}(A) = \{\emptyset\}$ has one element and $1 = 2^0$ showing that the theorem holds for $n = 0$.

Suppose by induction that $n \geq 1$ and the theorem is true for all nonnegative integers less than n. Let A be a set with n elements. Since $n \geq 1$, A is nonempty. So, it has an element $a \in A$. If we delete this element we get $B = A - \{a\}$ which is a set with $n - 1$ elements. By induction, B has exactly 2^{n-1} subsets.

For every subset S of A, $S \cap B$ is a subset of B. This is a 2-1 correspondence between subsets of A and subsets of B since, for every subset $T \subseteq B$, there are two subsets of A which correspond to T, namely T and $T \cup \{a\}$. So, A has twice as many subsets as B. So, A has

$$2 \cdot 2^{n-1} = 2^n$$

subsets proving that the theorem holds for all $n \geq 0$. □

Now I want to explain this proof in terms of the well-ordered property.

We know the theorem holds for $n = 0$ (by the base case in the proof above). Suppose by contradiction that the theorem is not true. Then there is a finite set A so that A has n elements but the number of subsets is not equal to 2^n. We also know n is positive since the theorem is true for $n = 0$. By the well-ordered property there is a smallest positive integer n for which there is such a counterexample. "Smallest" n means that the theorem holds for all sets with fewer than n elements. Take $a \in A$, $B = A - \{a\}$. Then B has 2^{n-1} subsets. So, A has 2^n subsets (by the proof above). This contradicts the assumption that A does not satisfy the theorem. This contradiction proves the theorem for all finite sets A.

There is famous joke showing a false use of strong induction:

Prove by induction that all horses have the same color.

If you have one horse then it has the same color as itself. So, the theorem holds for $n = 1$. Suppose that you have n horses and suppose by induction that the statement holds for $n - 1$. Remove one horse. Then the remaining horses have the same color by induction. Put it back and remove a different horse. Again, the remaining horses have the same color. So, they all have the same color. QED.

4.6. **worksheet.** (1) Show using the strong induction/well-ordered property that

$$1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1$$

1a) First take the case $n = 0$ and show it is true.

1b) Prove the statement ($P(n)$) assuming it is true for all smaller numbers (in particular for $n - 1$).

1c) What is the wording/logic if you are using strong induction? What is the logic if you are using the well-ordered property?

1) When $n = 4$ the LHS is equal to 1 (Note that there are $n + 1$ terms on the left side and the first terms is $2^0 = 1$.) The RHS is equal to $2^1 - 1 = 2 - 1 = 1$. So, the equation holds for $n = 0$.

1b) Now suppose by induction that $n \geq 1$ and the equation holds for all nonnegative numbers $< n$. Then

$$1 + 2 + 4 + 8 + \cdots + 2^n = 2^n - 1 + 2^n = 2 \cdot 2^n - 1 = 2^{n+1} - 1.$$

So, the equation holds for n and therefore, by induction it holds for all $n \geq 0$.

1c) The wording I gave is for strong induction. For the well-ordered property, I would have said: Suppose that the equation fails for some n. Then, by the well-ordered property, there is a smallest n for which it fails. But this implies that the equation holds for all smaller n. (Insert calculation above.) This gives a contradiction to the assumption that the equation does not hold. So, the equation must hold for all n.

(2) Show using the well-ordered property that there is no rational number n/m whose square is 3.

2a) Suppose that there are positive numbers n, m so that $n^2/m^2 = 3$. Express this as a property of the number n.

2b) Take the smallest n with this property and get a contradiction.

4.7. **Homework 4.** This homework is due on the Thursday after the break. Quiz 1 will be on the Wednesday after the break covering everything up through simple induction. More details will be posted. I will explain briefly the answers to HW4 in class on Monday and we will also review for the quiz.

In each problem start your answer by stating the assumptions and end it with a conclusion stating the result. In between, make sure each step is justified and explained.

(1) Prove by induction on n that

$$\sum_{i=1}^{n} 3i(i-1) = n^3 - n$$

(2) Write out the answers to the second problem in today's worksheet and hand it in as homework. (Show using the well-ordered property that there is no rational number n/m whose square is 3.)

(3) Take the sum:

$$\sum_{k=1}^{n} 5 \cdot 3^k$$

a) What is this sum for $n = 1, 2, 3$?

b) Find a formula for the sum and prove it by induction.

(4) Find the number of squares (with integer side lengths) inside a $3 \times n$ square and prove it using induction. For example, a 3×3 square has 14 squares: 9 little squares small 4 two-by-two squares and one big square. (Note: The same formula does not work for all n.)

(5) You have 3^n coins and one of them is heavier than the others. You also have a scale. Show that you can find the heavy coin in exactly n weighings.

(6) [3.49(b)] Determine for which positive integers n the following inequality holds and prove it by induction.

$$2^n \geq (n+1)^2$$

(7) (L-tiling) Show that R_n has an L-tiling if n is a power of 2. Look at the L-tiling problem in the book (p. 61). There is a complicated inductive solution for tiling R_n for arbitrary n. (R_n is the bottom right shape on p.61.) However, when n is a power of 2 ($n = 2^k$), there is an easy simple induction proof. Your job is to find this proof and explain it.

(8) [3.19] Prove by induction on n that

$$(\forall n \in \mathbb{N})(\forall x, y \in \mathbb{R}) x < y \Rightarrow x^{2k-1} < y^{2k-1}$$

4.8. **Worksheet.**

4.8.1. Draw a truth table for the statement $(p \vee \neg q) \wedge (p \Rightarrow r)$. Is this a tautology?

If A, B, C are sets and $P(x), Q(x), R(x)$ are the assertions $x \in A, x \in B, x \in C$ respectively, then write the set

$$D = \{x \in U \mid (P(x) \vee \neg Q(x)) \wedge (P(x) \Rightarrow R(x))\}$$

in terms of union, intersection, complements, etc using A, B, C. [Hint: Look at your truth table.]

Here is an answer suggested by one student:

First, convert to $\wedge, \vee, \neg$:

$$D = \{x \in U \mid (P(x) \vee \neg Q(x)) \wedge (\neg P(x) \vee R(x))\}$$

which becomes:

$$D = (A \cup B^c) \cap (A^c \cup C)$$

4.8.2. Is the following function bounded? increasing 1-1?

$$f : (-\infty, 0] \cup [1, \infty) \to \mathbb{R}$$

given by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \geq 1 \\ 2 - 3x & \text{otherwise} \end{cases}$$

This function is not bounded since it goes to ∞ as x goes to $-\infty$. It is not increasing since, e.g., $f(0) = 2 > f(1) = 1$. This function is 1-1 because it is decreasing. (To prove that f is a decreasing function take any $x < y$ in the domain. Then there are three possibilities: either $x < y \leq 0$, $1 \leq x < y$ or $x \leq 0$ and $y \geq 1$. In the first case, $-3x > -3y$ since multiplying by a negative number reverses the direction of the inequality, so $2 - 3x > 2 - 3y$. In the second case, $y > x > 0$ implies that $1/x > 1/y$. Finally, when $x \leq 0, y \geq 1$ we have $f(x) \geq 2$ and $f(y) \leq 1$. Therefore, in all three cases $f(x) > f(y)$ making f a decreasing function.)

Find a real valued function on the half open interval $(0, 1]$ which is not bounded, not increasing and not 1-1.

It easy to find functions with just one or two of these properties and we can just paste them together. For example:

$$f(x) = \begin{cases} 1/x & \text{if } 0 < x \leq 1/2 \\ 3 & \text{if } 1/2 < x \leq 1 \end{cases}$$

The function $1/x$ is unbounded and not increasing on the domain $(0, 1]$. But it is 1-1. So we needed another function. The function $f(x) = 3$ (a horizontal line of slope zero) is not 1-1 since it sends everything to one number (3).

4.8.3. Find for which positive integers n the following inequality holds and prove it by induction.

$$n^2 > 5n - 4$$

I claim that this is true for all $n \geq 5$. First, take $n = 5$. Then $5^2 = 25$ which is greater than $5 \cdot 5 - 4 = 21$. So, the inequality holds in the base case $n = 5$. Now suppose by induction that the inequality holds for n and $n \geq 5$. Then we need to show that $(n + 1)^2 > 5(n + 1) - 4$. But:

$$(n + 1)^2 = n^2 + 2n + 1$$

By induction, $n^2 > 5n - 4$. So,

$$n^2 + 2n + 1 > (5n - 4) + 2n + 1 = 7n - 3 = [5(n + 1) - 4] + 2n - 2$$

But, we also know that $n \geq 5$. So, $2n - 2 \geq 8$ making the last expression greater than $5(n + 1) - 4$. So

$$(n + 1)^2 > [5(n + 1) - 4] + 2n - 2 > 5(n + 1) - 4$$

which is what we wanted to show. So, the inequality holds for $n + 1$ if it holds for n. So, by induction, it holds for all $n \geq 5$.

4.8.4. Write the following statement in logical notation, find its negation and write the negation without using words of negation such as "not" or "no" or "without" or "nonnegative" or "nonpositive" or any symbol with a line though it ($\neq$, $\not\leq$, etc)

The image of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is contained in the set of positive real numbers.

This statement says that one set is contained in another set. If we negate this we would have the statement that the first set is not contained in the second set. (which says there exists an element of the first set which is not in the second).

In class I first rephrased the statement so that it is an implication. (The containment of one set in another is always an implication: $x \in A \Rightarrow x \in B$.) The statement rephrased was:

$$(\forall x \in \mathbb{R})x^2 > 0$$

The negation is

$$(\exists x \in \mathbb{R})x^2 \leq 0$$

In words this is:

There is a real number whose square is less than or equal to zero.

4.8.5. If a, b are positive real numbers so that $2a - b \leq 9$ then prove that either $a \leq 12$ or $b > 15$. Write your answer in complete sentences.

Suppose by contradiction that the conclusion does not hold. Then $a > 12$ and $b \leq 15$. This implies that $2a > 24$ and $-b \leq -15$. Adding these together we get

$$2a - b > 24 - 15 = 9$$

which contradicts the assumption that $2a - b \leq 9$. Therefore, the conclusion must be true and $2a - b \leq 9$ implies that $a \leq 12$ or $b > 15$.

One student said that he found a shorter proof. This proof was also intended to illustrate the concept of the *contrapositive* which is an important concept that I did not cover. The contrapositive of the statement $p \Rightarrow q$ is the statement $\neg q \Rightarrow \neg p$. This is equivalent to the original statement. What I proved was the contrapositive of the theorem since I assumed that the conclusion was not true and I deduced that the assumption is not true. (So, using the contrapositive is the same as proof by contradiction.) Students often confuse this with the *converse* which is the statement $q \Rightarrow p$ which is not equivalent to $p \Rightarrow q$, nor is it the negation of $p \Rightarrow q$. (But if you want to sound really smart, at the end of a math lecture you can ask the question: "Is the converse of your theorem true?")

5. Bijections and Cardinality

Two sets have the same number of elements if there is a 1-1 correspondence between the elements. For example, the set $P = \{A, B, C\}$ (Alice, Bob, Carl) has the same size as the set $H = \{165, 172, 195\}$ (height in cm). There is 1-1 correspondence given by the (height) function

$$h : P \to H$$

given by $h(A) = 195, h(B) = 165, h(C) = 172$.

Definition 5.1. A function $f : A \to B$ is a **bijection** if for every $b \in B$ there is a unique $a \in A$ so that $f(a) = b$.

Usually, we want to separate this definition into two parts:

(1) (existence) For each $b \in B$ *there is* an $a \in A$ so that $f(a) = b$. (We say f is **onto** is it has this property.)

(2) (uniqueness) This $a \in A$ is unique. So, if $a' \neq a$ then $f(a') \neq f(a) = b$. (In other words, f is **1-1**.)

Since each $b \in B$ gives a unique $a \in A$, we have another function $g(b) = a$ so that $f(a) = b$, or:

$$f(g(b)) = b$$

Theorem 5.2. *If $f : A \to B$ is a bijection then there is a unique function $g : B \to A$ so that $f(g(b)) = b$ for all $b \in B$ and $g(f(a)) = a$ for all $a \in A$. Furthermore, g is a bijection.*

Definition 5.3. Two sets A, B have the same **cardinality** and we write $|A| = |B|$ if there exists a bijection $f : A \to B$.

Definition 5.4. If $k \geq 0$ is a nonnegative integer, the cardinality of the set $[k] = \{1, 2, 3, \cdots, k\}$ is defined to be k. (When $k = 0$ this set is empty: $[0] := \{\ \} = \emptyset$.)

If A is a set with 4 elements then the definition says there is a bijection:

$$f : [4] = \{1, 2, 3, 4\} \to A$$

This gives 4 elements of A:

$$f(1), f(2), f(3), f(4)$$

The definition says:

(1) For each $a \in A$ there is a number $i \in [4]$ so that $a = f(i)$.

(2) If $i \neq j$ then $f(i) \neq f(j)$.

In different words, this says: The elements of A are $f(1), f(2), f(3), f(4)$ without repetition in this sequence.

Problem: Show that there is a bijection between the set of positive integers and the set of nonnegative integers:

$$\mathbb{Z}_{>0} = \mathbb{N} = \{1, 2, 3, 4, \cdots\}$$

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \cdots\}$$

This is easy to solve but we also need to write the proof. The bijection is:

$$f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{>0}$$

given by $f(n) = n + 1$.

Proof that this is a bijection. (1) We need to show that for every b in the second set there is an a in the first set so that $f(a) = b$. Since $b > 0$, we can take $a = b - 1 \geq 0$. Then $f(a) = a + 1 = (b - 1) + 1 = b$. So, f is onto. (It has the first property of a bijection.)

(2) We need to show that a is unique. So, suppose a' is another element of the first set and $a' \neq a = b - 1$. Then

$$f(a') = a\text{''} + 1 \neq a + 1 = b = f(a)$$

So, f is 1-1. Therefore, f is a bijection. □

Notice that we found a bijection between a set $\mathbb{Z}_{\geq 0}$ and a proper subset $\mathbb{Z}_{>0}$.

Definition 5.5. A set is called **infinite** if there is a bijection between the set and a proper subset.

The classical example is: You enter a movie theater which is full (every seat is occupied). But it has an infinite number of seats. So, you ask everyone to get up and move over to the next seat. Then you sit down!

The fact that the theater was full when you entered means that the people found a bijection:

$$S : \text{set of viewers} \to \text{set of seats}$$

This bijection might be called the "seat assignment" The fact that the theater is full means that S is onto. The fact that it is 1-1 means that each seat has only one person sitting in it.

Problem: Find a bijection between 3-nary sequences of length n and $[3^n]$ showing that the set has 3^n elements.

A 3-nary sequence means a sequence of digits 0,1,2. For example: 0211201 is a 3-nary sequence of length 7. We can write this as:

$$a_1 a_2 a_3 \cdots a_n$$

where each $a_i = 0, 1$ or 2.

The bijection is given by

$$f(a_1a_2\cdots a_n) = 1 + a_1 + 3a_2 + 9a_3 + \cdots + 3^{n-1}a_n$$

5.1. **Worksheet/Homework 5.** Due next Thursday, March 4.

(1) Show that there is a bijection between the set of positive integers and the set of all integers. The function is given by sending the even integers to the positive integers and the odd integers to 0 and the negative integers.

(a) Write down a formula for the mapping ("mapping" means "function" with the connotation that this is set theory and not a calculus type function).

$$f : \mathbb{N} = \{1, 2, 3, \cdots\} \to \mathbb{Z}$$

so that $f(2) = 1, f(4) = 2, \cdots$ and $f(1) = 0, f(3) = -1, f(5) = -2, \cdots$.

(b) Prove that this mapping is onto: Take an element of the second set $n \in \mathbb{Z}$ and consider two cases: $n > 0$ and $n \leq 0$.

(c) Prove that f is 1-1: Take another $a' \neq a$ and show that $f(a') \neq f(a)$.

(2) Suppose that a, b are fixed constants and $a \neq 0$. Then show that the function

$$f : \mathbb{R} \to \mathbb{R}$$

given by $f(x) = ax + b$ is a bijection.

(a) For each $y \in \mathbb{R}$ find a number x so that $f(x) = y$. (Solve the equation for x.)

(b) Show that no two distinct x give the same y.

(c) Conclude that the function is a bijection.

(3) Find a bijection from $(0, 1)$ to $\mathbb{R}$. Explain why you know it is a bijection but you don't have to write a rigorous proof.

(4) Write the following statement in logical notation: The set B has 5 elements. [Use the notation B^A for the set of all functions $f : A \to B$.] Write the negation of this logical statement. [It is not correct to assume that when two sets have different Cardinalities one set must be bigger than the other. This concept uses the Schroeder-Burnstein Theorem which we have not yet covered.]

(5) [This is 3.22] Show by induction on n that

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|$$

[Use the triangle inequality.]

(6) [This is 2.50] Do 2.50 (b),(c),(d). Here is the answer to (a):
a) Prove that $(A \cup B)^c = A^c \cap B^c$

Proof. We will show that each set is contained in the other. First, take any element $x \in (A \cup B)^c$. Then x is not contained in the union which means it is not true that $x \in A \cup B$. But $x \in A \cup B$ means $x \in A$ or $x \in B$. So, the negation is $x \notin A$ and $x \notin B$ which means $x \in A^c$ and $x \in B^c$ which is the same as saying $x \in A^c \cap B^c$.

Conversely, suppose that $x \in A^c \cap B^c$ which means $x \in A^c$ and $x \in B^c$. Then we need to show that $x \in (A \cup B)^c$ in other words that x is not in $A \cup B$. Suppose by contradiction that this is not true. Then $x \in A \cup B$. So either $x \in A$ or $x \in B$ but each of these is impossible since $x \notin A$ and $x \notin B$. Therefore, it must be true that $x \in (A \cup B)^c$.

Since the two sets are contained in each other, they are equal. $\square$

5.2. **more bijections.** Recall that a *bijection* is a mapping

$$f : A \to B$$

which is 1-1 and onto. In the book these properties are called **injective** (1-1) and **surjective** (onto). But "1-1" and "onto" use less chalk. (Also the words "bijective", "injective", "surjective" all sound the same.) These are also called "monomorphisms" and "epimorphisms" which are sometimes abbreviated "mono" and "epi".

Problem: Find a bijection

$$f : [0, \infty) \to (0, \infty)$$

Solution: We just need to push the number 0 into the rest of the set and tell everyone to move over:

$$f(x) = \begin{cases} x + 1 & \text{if } x \text{ is an integer} \\ x & \text{otherwise} \end{cases}$$

Problem: Find a bijection between the set T of 3-nary sequences of length n and the set

$$S = \{0, 1, 2, 3, 4, \cdots, 3^n - 1\}$$

showing that the set T has 3^n elements.

A 3-nary sequence means a sequence of digits 0,1,2. For example: 0211201 is a 3-nary sequence of length 7. We can write this as:

$$a_0 a_1 a_2 \cdots a_{n-1}$$

where each $a_i = 0, 1$ or 2.

The bijection is given by

$$f(a_0 a_1 \cdots a_{n-1}) = a_0 + 3a_1 + 9a_2 + \cdots + 3^{n-1} a_{n-1}$$

To prove that f is a bijection, we will use the "pigeonhole principle" which says that "If there are n pigeons in n pigeonhole and each pigeonhole is occupied by at least one pigeon then every pigeonhole has exactly one pigeon" In other words, there is a bijection between pigeonholes and pigeons. We will prove this later. Right now I want to use it in its mathematical version:

Theorem 5.6 (Pigeonhole principle)**.** *If A, B are finite sets of the same size and $f : A \to B$ is surjective (onto) then f is a bijection.*

To use this: We have a function $f : T \to S$ which we want to show is a bijection. The two sets are finite with the same size. So, we just need to show that f is onto.

This is by induction on n. If $n = 1$ then T and S are the same set and f is the **identity mapping** $f(a) = a$.

Suppose by strong induction that $n \geq 2$ and the statement holds for smaller numbers. Then take any nonnegative integer $m < 3^n$. When we divide by 3 we get a remainder r of 0,1 or 2. Let $a_0 = r$ be this remainder. Then

$$m = 3q + r$$

and $q = (m - r)/3$ is a nonnegative integer $< 3^{n-1}$. So, by strong induction, there is a 3-nary sequence $b_0 b_1 \cdots b_{n-2}$ of length $n - 1$ so that

$$q = b_0 + 3b_1 + \cdots + 3^{n-2} b_{n-2}$$

But then

$$\begin{aligned} m = 3q + r &= r + 3b_0 + 9b_1 + \cdots + 3^{n-1} b_{n-2} \\ &= f(r b_0 b_1 \cdots b_{n-2}) \end{aligned}$$

So, f is onto and therefore a bijection by the pigeonhole principle.

Example 5.7. The **weight problem**: Suppose that you have a scale which balances but does not measure. Suppose you have k weights:

$$1, 3, 9, \cdots, 3^{k-1}$$

a) Show by induction on k that the total weight is

$$\sum_{i=1}^{k-1} 3^i = 1 + 3 + \cdots + 3^{k-1} = \frac{3^k - 1}{2}$$

b) Show that, using these weights, we can weight any integer amount between 1 and $\frac{3^k-1}{2}$.

This is a bijection problem. We make a set A which will represent all possible arrangements of weights. If you put an object of unknown weight on your scale you can put your weights on the same side or the other side. For example, if your object weights 7 oz then you put it on the right side of the scale together with the 3oz weight and you put the 9 and 1 oz weights on the left side. Then:

$$9 + 1 = 3 + 7$$

so, the scale balances. You label this arrangement

$$(1, -1, 1, 0, 0, \cdots, 0)$$

where 1 means you put the weight on the left side, -1 means put the weight on the right side and 0 means you don't use the weight. The formula is:

$$\mathbf{b} = (b_0, b_1, b_2, \cdots, b_{k-1})$$

represents the arrangement where the weight 3^i is place on the left side if $b_i = 1$ and 3^i is placed on the right side (next to the unknown weight) if $b_i = -1$. When you do this, the scale balances if your unknown weight is equal to

$$f(\mathbf{b}) = \sum_{i=0}^{k-1} b_i 3^i$$

Let B be the set of all sequences $(b_0, b_1, b_2, \cdots, b_{k-1})$ where each b_i is 0,1 or -1. Since there are k coordinates for this vector and each coordinate has three possibilities, this set has size (cardinality)

$$|B| = 3^k$$

Theorem 5.8. *The function f gives a bijection from this set B to the set of integers*

$$A = \left\{ n \in \mathbb{Z} \,|\, |n| \le \frac{3^k - 1}{2} \right\}$$

In order to prove that a function $f : B \to A$ is a bijection, the fastest way is to find another function $g : A \to B$ so that

(1) $g(f(b)) = b$ for all $b \in B$ and
(2) $f(g(a)) = a$ for all $a \in A$.

(more on this later)

5.3. **worksheet.** (1) Show that every integer between 1 and $2^n - 1$ can be written as a sum of distinct powers of 2 by making a bijection between the set of all binary sequences of length n and the integers from 0 to $2^n - 1$.

(2) Find a bijection between the sets $\mathbb{R}$ and $(-\infty, 0] \cup [1, \infty)$

(3) Find a bijection between the set of all functions $[n] \to [3]$ and the set of all integers from 1 to 3^n. [hint: subtract 1]

5.4. **Properties of composition.** In calculus you learned the derivative of a composition of functions. We will study the set theoretic properties of compositions.

Definition 5.9. If f and g are functions so that the target set of f is equal to the domain of g:

$$f : A \to B \qquad g : B \to C$$

then the **composition** $g \circ f$ is defined to be the function

$$g \circ f : A \to C$$

given by $(g \circ f)(a) = g(f(a))$ for every $a \in A$.

Theorem 5.10. *The composition of two surjective functions is surjective.*

Proof. Suppose that $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$ are onto. Then we want to show that $g \circ f : A \to C$ is onto. So, take any $c \in C$. Since g is onto, there is an element $b \in B$ so that $g(b) = c$. Since f is onto, there is an element $a \in A$ so that $f(a) = b$. But then

$$(g \circ f)(a) = g(f(a)) = g(b) = c$$

Since we stated with an arbitrary element $c \in C$ and we found an element $a \in A$ which maps onto it, we have shown that $g \circ f$ is surjective. □

Recall that there are synonyms:
(adjective form) onto, surjective
(noun form) surjection, epimorphism
Also for 1-1 is the same as *injective* with noun forms *injection* and *monomorphism*.

Problem: Find an example of two functions f, g so that $f \circ g$ is surjective but g is not surjective.

Theorem 5.11. *The composition of injective maps is injective.*

Proof. Let a_1, a_2 be distinct elements of A. Since f is injective, $b_1 = f(a_1) \neq b_2 = f(a_2)$. Since g is injective, this implies that $g(b_1) \neq g(b_2)$. So

$$(g \circ f)(a_1) = g(f(a_1)) = g(b_1) \neq g(b_2) = g(f(a_2)) = (g \circ f)(a_2)$$

Therefore, $g \circ f$ is 1-1. □

Problem: Show that if $g \circ f$ is injective then f is necessarily injective. Give an example to show that g need to be injective.

Definition 5.12. The **identity function** on a set S is the function $id : S \to S$ given by $id(x) = x$. Sometimes we write id_S in case there are lots of identity functions.

Theorem 5.13. *A function $f : A \to B$ is a bijection if and only if there is a function $g : B \to A$ so that $f \circ g = id_B$ and $g \circ f = id_A$.*

5.5. **Composition of functions worksheet.** (1) Give an example of two functions $f : A \to B$ and $g : B \to C$ so that the composition $g \circ f : A \to C$ is a bijection but f, g are not bijections.

[Example means explicit example, you could say $A = \{1, 2\}$ or you can say $A = \{a, b\}$ but in the second case you also have to stipulate that $a \neq b$.]

(2) If $g \circ f$ is a bijection, then show that g is onto and f is 1-1.

(2a) Take $c \in C$. You want to find some $b \in B$ so that $g(b) = c$. What can you conclude from the fact that $g \circ f$ is a bijection? How does that help?

(2c) To show that f is 1-1 you take two distinct elements of A. Give them names. Give names for their images in B and C. What do you know about these 6 elements? What are you supposed to prove about theses elements?

(3) Write down an explicit 1-1 correspondence between the subsets of $[n]$ and functions $g : [n] \to \{0, 1\}$.

Note: (the syntax of 1-1 correspondence) A *bijection* between sets $f : A \to B$ is called a 1-1 correspondence between the elements of A and the elements of B. So, the question is asking for a bijection between the two sets:

(a) $\mathcal{P}([n])$, the set of all subsets of $[n]$ and

(b) $\{0, 1\}^{[n]}$ the set of all functions $g : [n] \to \{0, 1\}$.

The answer associates to each subset $S \subseteq [n]$ its **characteristic function** $\chi_S : [n] \to \{0, 1\}$ defined by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

What is the inverse function? In other words, if I give you the function g, find the set S so that $g = \chi_S$.

5.6. **countable sets.** Here is an outline of countable sets.

Definition 5.14. set S is **countable** if either S is empty or there is an surjection $\mathbb{N} \to S$. A set S is **countably infinite** if it has the cardinality of $\mathbb{N}$, i.e., if there is a bijection $\mathbb{N} \to S$.

Theorem 5.15. *Countable sets are either finite or countably infinite.*

Theorem 5.16 (Cantor)**.** *The set of real numbers is uncountable.*

(Proof using Cantor's diagonalization argument)

Theorem 5.17. *Every subset S of the natural numbers is countable.*

Proof. Either S is finite or infinite. If S is infinite define the surjection $f : \mathbb{N} \to S$ by letting $f(n)$ be the smallest element of S which is $\geq n$. □

Theorem 5.18. *The set of rational numbers is countable.*

Proof. A surjection $f : \mathbb{N} \to \mathbb{Q}$ is given as follows. For any $n \in \mathbb{N}$ write the prime factorization of n:

$$n = 2^a 3^b 5^c \cdots$$

Define $f(n)$ to be the fraction $f(n) = b/(c+1)$ if n is odd and $f(n) = -b/(c+1)$ if n is even. Then any nonnegative fraction x/y is equal to $f(3^x 5^{y-1})$ and $-x/y = f(2 \cdot 3^x 5^{y-1})$. □

5.7. **worksheet.** (1) Show that every subset of a countable set is countable.

(1a) Choose symbols for your two sets. Divide into two cases depending on whether the countable set that you start with is finite or infinite.

(1b) If your set is infinite then use the bijection with $\mathbb{N}$ to complete the proof.

(1c) You can state without proof that every subset of a finite set is finite.

(2) Show that the union of two countable sets is countable.

(2a) Choose notation for your sets and for the surjective functions which are given by the definition of countability.

(2b) Divide the natural numbers into odd and even numbers, then map the even numbers onto the first set and the odd numbers onto the second set.

(2c) Show that your function is onto.

5.8. **Review 2a.** This is the first review sheet of Quiz 2. The topics for Quiz 2 are induction and strong induction and bijections and cardinality. Countability will not be on Quiz 2 but will be on HW 6.

(1) Calculate the sum

$$\sum_{k=1}^{n} \frac{3k-2}{5}$$

(2) Prove that

$$\sum_{k=1}^{n} 3^k = \frac{3^{n+1}-1}{2}$$

(3) If $f : A \to B$ is surjective but not injective and $g : B \to C$ is injective but not surjective then does it follow that $g \circ f$ is a bijection? Does it follow that $g \circ f$ is not a bijection?

(4) Given a function $f : A \to B$, write in logical notation the statement that f is injective but not surjective.

(5) Find a function $f : [0, 1) \to \mathbb{R}$ which is surjective but not injective.

(6) You have three kinds of coins that you use as weights. They weight 3gm, 5gm, 6gm. Show that you can use your coins on one side of a scale to balance any integer number of weights ≥ 8 on the other side of the scale using stong induction.

(7) Given an example of a function $f : A \to A$ which is surjective but not 1-1.

5.9. **Review 2b.** (1) Calculate the following sum

$$\sum_{k=2}^{n} 5^k$$

(2a) Find a 1-1 function from $\mathbb{Z}$ to the set of rational numbers between 0 and 1.

(2b) Find a surjective function from $\mathbb{Q}$ to $\mathbb{Z}$.

(3) Determine the set of all integers n so that $2n^3 - 3n > |n|$ and prove it by induction on n.

(4) Write the statement "S has a subset with 5 elements" in logical notation. What is the negation of this statement?

(5) Where is the flaw in the proof by induction that all horses have the same color?

5.10. **Homework 6.** Due Thursday, March 18. The first 5 problems are from the worksheets. Look there for detailed instructions and hints.

(1) Write down an explicit 1-1 correspondence between the subsets of $[n]$ and functions $g : [n] \to \{0, 1\}$.

(2) Show that the union of two countable sets is countable.

(3) If $g \circ f$ is a bijection, then show that g is onto and f is 1-1.

(4) Find a bijection between the sets $\mathbb{R}$ and $(-\infty, 0] \cup [1, \infty)$ and prove that your mapping is a bijection. [Hint: Write $\mathbb{R}$ as a union $\mathbb{R} = (-\infty, 0] \cup (0, \infty)$. The easiest method to prove your map is a bijection is to find the inverse mapping g and show $f \circ g, g \circ f$ are both identity mappings on their respective domains.]

(5) Find a bijection between the set of all functions $[n] \to [3]$ and the set of all integers from 1 to 3^n and prove that your mapping is a bijection. [You may assume without proof that the set B^A of all mappings $f : A \to B$ has exactly m^n elements if $|A| = n$ and $|B| = m$ are finite. (with the indeterminate $0^0 = 1$). So, the pigeonhole principle applies.]

(6) Find an example of three sets A, B, C and three mappings $f : A \to B, g : B \to C, h : C \to A$ so that the composition

$$h \circ g \circ f : A \to A$$

is the identity mapping on A but

$$g \circ f \circ h : C \to C$$

is not the identity mapping on C.

Make sure that you give specific sets A, B, C and functions f, g, h. These can be given by formulas or by a list of values (when the sets are finite). Show that the functions have the required properties by doing the appropriate calculations.

(7) Write in words the following two statements. Here $(\exists S)$ means "there exists a set S"

$$(\exists S)(\forall T)(\forall f \in T^S)(\exists g \in S^T)(\forall x \in S)g(f(x)) = x$$
$$(\forall S)(\exists T)(\forall f \in T^S)(\exists g \in S^T)(\forall x \in S)g(f(x)) = x$$

Give one example to show that the second statement is not true. (You don't have to prove the first statement.)

(8) Write the following statement in logical notation (similar to problem (7)). For any surjective mapping $f : S \to T$ between any two sets there is a mapping $g : T \to S$ so that $f \circ g : T \to T$ is the identity mapping on T. Then write its negation in symbols and in words. [This is one form of the Axiom of Choice. It is impossible to prove or disprove this statement. So, don't try!]

6. Graph theory

6.1. **worksheet.** (1) Combinations. If S is a set with n elements then the number of subsets of n with exactly k is given by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Let $S = \{a, b, c, d\}$ be a set with 4 elements. How many subsets of S are there with 2 elements? List all of them.

(2) In the following graph, label the vertices a, b, c, d and the edges $e_1, e_2, \cdots$. What is the set $\mathcal{P}_2(V)$? Find the mapping

$$h : E \to \mathcal{P}_2(V)$$

which sends each edge to its set of incident vertices.

(3) Draw the graph corresponding to the following data:

$$V = \{1, 2, 3, 4, 5, 6\}$$

$$E = \{a, b, c, d, e\}$$

$$h(a) = \{1, 2\},\ h(b) = \{2, 3\},\ h(c) = \{2, 4\},\ h(d) = \{4, 5\},\ h(e) = \{5, 6\}$$

6.2. **basic definitions.** I usually call graphs Γ but the book calls graphs G. So, I will use both. There are also two notations for edges. In standard terminology, the graphs that we consider are finite graphs with *multiple edges* and no *loops*.

Definition 6.1. A **graph** Γ is a triple $\Gamma = (V, E, h)$ where

(1) V is a finite set called the **vertex set**. The elements of V are called **vertices**.
(2) E is a finite set called the **edge set**. The elements of E are the **edges** of the graph.
(3) h is a mapping
$$h : E \to \mathcal{P}_2(V)$$
from the edge set E to the set $\mathcal{P}_2(V)$ of all 2-element subsets of V. This is the **incidence function** or *incidence map*.

An edge e is **incident** to its two **endpoints** a, b (if $h(e) = \{a, b\}$. Two vertices a, b are **adjacent** if there is one edge incident to both.

The interpretation of this mathematical data is that the vertices are point and the edges are line segments connecting the two incident vertices. (If $h(e) = \{a, b\}$ then the vertices a, b are said to be **incident** to e.) When the incidence function is not injective we have a *multiple edge*. For example if $h(e_1) = h(e_2) = h(e_3) = \{a, b\}$ then we have three edges incident to a and b. A *loop* is an edge with both endpoints at the same vertex: $h(e) = \{a, a\} = \{a\}$. The definition above excludes this since $h(e)$ must have two elements. The definition in the book is ambiguous about loops.

There is a second definition for *simple graphs*. When the function h is injective, the graph is called **simple**. In that case, we assume that E is a subset of $\mathcal{P}_2(V)$ and $h : E \hookrightarrow \mathcal{P}_2(V)$ is the inclusion mapping and we write the shorthand $e = ab$ instead of $h(e) = \{a, b\}$. When there are multiple edges we can still use this shorthand notation but we would write $h(e) = ab$ and, e.g., $h(e_1) = ab = h(e_2)$.

Theorem 6.2. *If S is a finite set with n elements, the number of k element subsets of S is given by the binomial coefficient*
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$
where $n! = n(n-1)(n-2)\cdots 2 \cdot 1$ is defined recursively by

(1) $0! = 1$
(2) $n! = n(n-1)!$ *for* $n \geq 1$.

I proved only the case $k = 2$ in class. We will look at the general case later.

Proof. By induction on n. If $n = 1$ then the set S has 0 subsets with two elements. The formula gives $n(n-1)/2 = 1(0)/2 = 0$. So the theorem holds in this case.

Suppose that the formula holds for n. Then we want to show that it holds for $n+1$. So, suppose S is a set with $n+1$ elements:

$$S = \{a_1, a_2, \cdots, a_n, a_{n+1}\}$$

Suppose that A is a subset of S with exactly two elements. Then there are two possibilities:

Case 1: $a_{n+1} \in A$

Case 2: $a_{n+1} \notin A$.

In the second case A is a subset of $\{a_1, \cdot, a_n\}$. We know by induction that there are $n(n-1)/2$ subsets A of this kind with two elements. In the first case, A consists of a_{n+1} and one of the elements $a_1, \cdots, a_n$. So, there are exactly n subsets A in case 1. Putting these together we get a total of

$$\frac{n(n-1)}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \binom{n+1}{2}$$

subsets with 2 elements in S. This proves the formula for $n+1$. By induction it holds for all $n \geq 1$. □

Definition 6.3. The **degree** of a vertex v is defined to be the number of incident edges:

$$deg(v) = d(v) = |e \in E : v \in h(e)|$$

Another word for degree is **valence** especially in adjective form. For example, **trivalent** graph means every vertex has degree 3.

Problem Show that the sum of the degrees of all the vertices is equal to twice the number of edges:

$$\sum_{v \in V} deg(v) = 2|E|$$

Definition 6.4. A **trail** in a graph $\Gamma = (V, E, h)$ is defined to be a sequence of vertices and edges: *vertex-edge-vertex-edge-etc-edge-vertex*:

$$v_0, e_1, v_1, e_2, \cdots, e_n, v_n$$

so that the edges e_i are distinct. The **length** of a trail is the number of edges.

Note that a trail has a starting point, and end point and a direction. It starts at v_0 and ends at v_n.

For example,

$$v, e_1, a, e_2, b, e_3, v, e_4, w$$

is a trail of length 4 which starts at vertex v and ends at vertex w. It visits the same vertex v twice but does not use the same edge twice. If the trail begins and ends at the same vertex, it is said to be **closed**.

Definition 6.5. The **complete graph** with n vertices is the simple graph K_n with vertex set $[n] = \{1, 2, \cdots, n\}$ and edge set $E = \mathcal{P}_2([n])$. In other words, every pair of edges is connected by one edge. Sometimes we write $V = \{v_1, v_2, \cdots, v_n\}$ (to emphasize that these are vertices).

6.3. **worksheet.** (1) Draw the complete graph with 6 vertices, label the vertices and find two different closed trails of length 6 which go through all the vertices. Write down these trails in the notation:

$$v_1, v_1v_2, v_2, \cdots$$

(2) Take the simple graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_4, v_3v_4\}$. Draw it and find the longest possible trail. Is there more than one?

(3) Can you prove the following: If a maximal trail is not close, the end points have odd degree.

As we discussed in class, if a vertex has even degree, you cannot get stuck at the vertex unless you start at that vertex. The reason is that whenever you are at the vertex you have entered it one more time than you left it. So, you use an odd number of the roads to the vertex. More on this later.

6.4. **review 3.** Quiz 3 is postponed until Thursday, April 15.
HW 7 will be due Thursday, April 8.
HW 8 will be due Thursday, April 22
Quiz 4 on April 29.
HW 9 due the last day of class (May 5).

Here are the problems that I prepared for review for Quiz 3. But there will be more topics on that quiz.

(1) What is the difference between a countable set and a countably infinite set?
(2) What is the precise statement of the theorem that Cantor proved using the diagonalization argument?
(3) How many edges does a complete graph on 6 vertices have?
(4) Draw a trivalent simple graph with 6 vertices.
(5) What is the definition of a simple graph? Include the definition of a graph in your definition.

More topics for quiz 3: (things we haven't covered yet)

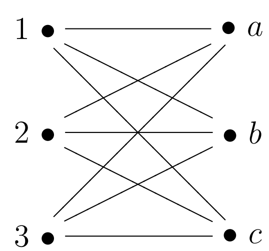
(1) Find all connected bipartite graphs with two vertices in each part. Which ones are isomorphic?
(2) A bivalent connected graph has exactly two Eulerian trails starting at each vertex. Why?
(3) Suppose that G is a connected graph which is 4-valent (every vertex has degree 4). Show that it has at least 8 Eulerian trails starting at each vertex. (There are 4 possibilities for the first step and at least 2 for the second.)

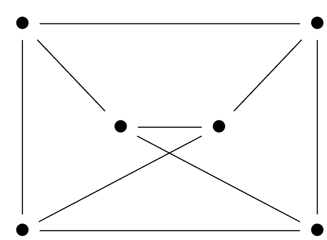
6.5. **worksheet.** (1) An **isomorphism** between two graphs $G = (V(G), E(G), h_G)$ and $H = (V(H), E(H), h_H)$ is a pair of bijections

$$f_0 : V(G) \to V(H), \qquad f_1 : E(G) \to E(H)$$

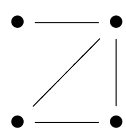
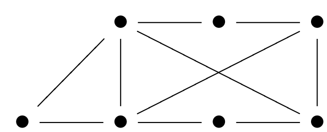
so that corresponding edges are incident to corresponding vertices. In equations: $h_H(f_1(e)) = f_0(h_G(e))$ for all $e \in E(G)$. When the graphs are simple graphs, an isomorphism $f : G \to H$ is given by just the vertex bijection $f_0 : V(G) \to V(H)$. It needs to satisfy the conditions that vertices v, w in G are adjacent if and only if $f_0(v), f_0(w)$ are adjacent in H.

Write down an isomorphism between the following graphs.

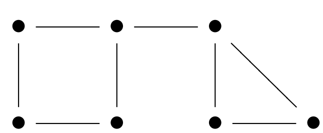
$G =$ 

$H =$ 

(2) Find all subgraphs of H which are isomorphic to G:

$G =$  $H =$ 

(3) A graph is **connected** if for any two vertices v, w there is a trail from v to w. A connected graph with n vertices and $n - 1$ edges is called a **tree**. Any connected graph with n vertices contains a tree with n vertices. This is called a **spanning tree**. Find all possible spanning trees in the following graph.



6.6. **topics covered so far.** Two graphs are *isomorphic* if they are the "same graph relabeled" A rigorous definition in on the worksheet.

Definition 6.6. A **bipartite graph** is a graph in which the vertex set can be separated into two disjoint subsets $V = A \coprod B$ so that no two vertices in the same part are adjacent.

A *tree* is a connected graph which has the fewest number of edges needed to connect the graph which is $n-1$ where n is the number of vertices. A tree (with at least two vertices) must have at least two leaves. In this book a *leaf* is defined to be a vertex of degree 1. Each leaf is incident to exactly one edge. Some people call this edge a "leaf."

A **cycle** is a graph having n vertices $v_1, \cdots, v_n$ and n edges $e_1, \cdots, e_n$ so that

$$h(e_i) = \{v_{i-1}, v_i\}$$

for $i = 1, \cdots, n$ where we define $v_0 = v_n$.

6.7. **Homework 7.** Due Thursday, April 8. There are 8 questions. Some of the topics in these questions are: degrees, Eulerian graphs, trees, numbers of cycles, leaves, perfect matchings, bipartite graphs.

Answer the questions using complete sentences. Give proof or explanations for all statements. For example, in 11.12, if you say that two of the graphs are not isomorphic, explain why. If two are isomorphic, give the isomorphism and explain why it is an isomorphism.

Chap 11, page 228, # 11.9, 12, 19(a) [Count the number of elements of $\mathcal{P}_2([n])$ in two different ways.], 23, 29, 30,

Variant of 11.3: For any integer $n \geq 2$ and any $k \geq n-1$ show that there is an Eulerian graph with n vertices and k edges.

Variant of 11.37: How many perfect matchings does $K_{n,n}$ have? How many 6 cycles does $K_{n,n}$ have?

6.8. **problems.** An **Eulerian trail** is a trail which goes through every edge exactly once.

(1) If a graph has an Eulerian trail prove that it has at most two vertices of odd degree.

(2) Theorem: A tree is connected graph without cycles.

Prove one part of this theorem: A tree has no cycles. [Hint: by contradiction. If it has a cycle, remove one edge.]

(3) The **chromatic number** of a graph is the smallest number of colors needed to color all the vertices so that no two adjacent vertices have the same color. If the chromatic number of Γ is k then the graph is k-partite. Conversely, if Γ is k-partite then the chromatic number is *k or less.*

Find a 4 partite graph whose chromatic number is 2.

6.9. **Problems.** (1) Suppose that G is a bipartite graph with parts A, B. If every vertex has degree 3 prove that $|A| = |B|$.

[Find a combinatorial proof: Count the number of edges in G in two different ways.]

(2) [Similar to HW7, 11.19(a)] Find a combinatorial proof of the formula:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

by counting the number of subsets of $[n+1]$ with k elements in two different ways. [Assume the formula given in class without a complete proof: $\binom{n}{k}$ is the number of k element subsets of a set with n elements.]

(3) A graph has the property that every vertex has degree 5. Show that there are an even number of vertices. [Assume that the number of vertices is odd and get a contradiction.]

6.10. **theorems.**

Lemma 6.7. *Every tree is bipartite and there are exactly two ways to color it with two colors.*

Theorem 6.8. *A graph Γ is bipartite if and only if it has no odd cycles.*

Proof. If Γ is bipartite, it is 2 colorable. So it cannot have an odd cycle since, the colors on vertices in a cycle must alternate. Conversely, suppose that Γ has no odd cycles. Then, each component of Γ has a spanning tree which is 2-colorable. Each additional edge (called a **chord**) must connect two vertices of oppose colors since each chord forms an even cycle with the tree. So, the coloring of the spanning trees gives a coloring of the graph. □

Lemma 6.9. *The sum of the degrees of the vertices in a graph is always even.*

Theorem 6.10. *A graph Γ is Eulerian if and only if it is connected and either*

(1) *Γ has exactly two vertices of odd degree.*
(2) *all vertices of Γ have even degree*

In the first case, there is an Eulerian trail connecting the two vertices of odd degree, in the second case there is a closed Eulerian trail starting at any point.

Here is Pascal's triangle which illustrates problem (2):

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Each entry is the sum of the two above it:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

So, for example:

$$
\begin{array}{ccccc}
\binom{5}{3} & = & \binom{4}{3} & + & \binom{4}{2} \\
10 & = & 6 & + & 4
\end{array}
$$

Theorem 6.11. *A graph* Γ *is Eulerian if and only if it is connected and either*

(1) Γ *has exactly two vertices of odd degree.*
(2) *all vertices of* Γ *have even degree*

In the first case, there is an Eulerian trail connecting the two vertices of odd degree, in the second case there is a closed Eulerian trail starting at any point.

Proof. If Γ is Eulerian, it has an Eulerian trail. If the trail is closed, every vertex has even degree since the trail enters each point the same number of times that it leaves. If the starting and ending points of the Eulerian trail are distinct then those two vertices have odd degree and the other vertices have even degree. So, one of the two conditions holds for any Eulerian graph.

Conversely, suppose that Γ satisfies one of the two conditions. To prove that Γ has an Eulerian trail we give an algorithm for finding this trail. Start at a vertex v of odd degree (or start at any vertex) if all vertices have even degree. Take the longest trail that you can starting at v. (Keep going until you get stuck.)

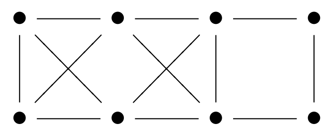
In case 1 you will end up at vertex w, the other vertex of odd degree. In case 2 you will end up back at vertex v. (Why is that?)

You get stuck because you have used up all the edges incident to the final vertex of your trail. If this is an Eulerian trail, you have succeeded and you stop. If this is not an Eulerian trail, you look at the edges that you used and the edges which are left over. Let H_1 be the graph made up of the edges which and vertices that you visited on your trail. Let H_2 be the graph made up of all the other edges and all their incident vertices.

Since Γ was connected, the graphs H_1 and H_2 share at least one vertex, call it x. All vertices in H_2 have even degree. (Why?) So, starting at the vertex x and taking the longest trail T_0 in H_2, we end up at the same vertex x.

Now "paste" together the two trails. Take your first trail, T. Then T starts at v and visits all vertices in H_1. In particular T visits vertex x at least once. So, $T = T_1T_2$ where T_1 is a trail from v to x and T_2 is a trail from x to w. Now insert the trail T_0 to get a longer trail $T_1T_0T_2$ starting at v and ending at w. If this new trail is not an Eulerian trail, repeat this process. Since the number of edges in H_2 keeps decreasing, this process will eventually stop when H_2 has no edges and your trail is an Eulerian trail. □

6.11. **worksheet.** (1) Apply the algorithm to find an Eulerian trail in the following graph. Make the process longer by making inefficient moves. What are the subgraphs H_1 and H_2 at each stage?

$G =$ 

(2) What is the definition of a trail? Make sure to use the correct syntax for mathematical definitions:

(a) Describe the setting (We have a graph and we are going to define something related to that graph.)

(b) A trail is an object satisfying conditions. What is the object? What kind of thing is it? Is it a function? a set? a sequence of sets?

(c) What are the conditions?

7. Recurrence relations

7.1. **basic definitions.** Notation: By a **sequence** we mean an infinite sequence:

$$a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \cdots$$

The sequence is denoted $\langle a \rangle$. The a_i are elements of some set S. And strictly speaking the sequence is a function

$$a : \mathbb{Z}_{\geq 0} \to S$$

For example, $a_n = 2n + 1$ is a formula for the sequence of integers

$$\langle a \rangle = (1, 3, 5, 7, 9, 11, \cdots)$$

Definition 7.1. A **recurrence relation** of **order** k is an expression or formula which, for $n \geq k$ gives a_n in terms of the quantities n and the previous k terms of the sequence which are:

$$a_{n-k}, \cdots, a_{n-2}, a_{n-1}$$

Example 7.2. For example, the famous Fibonacci sequence:

$$\langle a \rangle = (0, 1, 1, 2, 3, 5, 8, 13, \cdots)$$

is given by the second order recurrence:

$$a_n = a_{n-1} + a_{n-2}$$

Since $k = 2$, this formula only holds for $n \geq 2$.

$$a_0 = 1, a_1 = 1$$

are not given by the recurrence relation.

Example 7.3. The odd number sequence $a_n = 2n+1$ can be given by a recurrence of order 1:

$$a_n = a_{n-1} + 2$$

starting with $a_0 = 1$. But the same recurrence relation gives the even numbers if we start with $a_0 = 2$. So, in this example, we have two sequences satisfying the same recurrence.

Definition 7.4. If a sequence $\langle a \rangle = (a_0, a_1, \cdots)$ satisfies a recurrence relation of order k, the values of $a_0, a_1, \cdots, a_{k-1}$ are called the **initial values** of the sequence.

Theorem 7.5. *Given any recurrence relation of order k, the initial values of a sequence can be chosen arbitrarily and the rest of the sequence will be uniquely determined.*

7.2. **problems.**

Example 7.6. Draw 9 straight lines which are not parallel (so every pair of lines meets at a point) and so that no three lines go through the same point. The lines divide the plane into regions. How many regions do you get? Write down a recurrence relation for the number of regions you get with n lines.

The way to solve problems like this is to first do examples with small numbers. Take $n = 0, 1, 2, 3, 4, 5$ and count the number of regions. (With 0 lines you get $a_0 = 1$ region, with 1 line you get $a_1 = 2$ regions, etc.) You should see a pattern, especially for the difference between the numbers.

Calculations gave:

$$\begin{array}{cccccc} a_0, & a_1, & a_2, & a_3, & a_4, & a_5 \\ 1, & 2, & 4, & 7, & 11, & 16, \end{array}$$

with differences:

$$1, \quad 2, \quad 3, \quad 4, \quad 5,$$

So, the recurrence is

$$a_n = a_{n-1} + n$$

This has order $k = 1$. The proof that this holds is: The n-th line meets each of the previous $n-1$ lines cutting each of the regions bounded by those line into two regions. The number of regions is n since the strip around the n-th line is cut $n - 1$ times by these lines.

Example 7.7. Find all possible real number solutions to the first order recurrence:

$$a_n = a_{n-1} + 2$$

Here $k = 1$. So, the first value a_0 can be chosen arbitrarily. So, set it equal to $a_0 = x$. Next, what is a_1? a_2? a_n? Congratulations! You solved the problem.

This was easy:

$$a_n = x + 2$$

where x is any real number.

Example 7.8. Write down a recurrence relation for the numbers $a_n = n!$. What are the initial values? What is the order of the recurrence?

The recurrence is $a_n = n \cdot a_{n-1}$ this has order $k = 1$ with initial value $a_0 = 1$. (This is given by the definition $0! = 1$.)

7.3. **solution of linear 1st order recurrence.** I also had time to explain the solution of a typical first order linear recurrence.

Example 7.9. Find the general solution of the recursion:

$$a_n = 2a_{n-1} + 1$$

Since $k = 1$, we can start with any value of a_0 and the rest of the sequence will be determined. One amusing value is

$$a_0 = -1$$
$$a_1 = 2a_0 + 1 = 2(-1) + 1 = -1$$
$$a_2 = -1, a_3 = -1, \cdots, a_n = -1$$

This is called a **particular solution** to the recurrence. The theory, which I will explain later says we should combine this with a solution of:

The **associated homogeneous recurrence** which is:

$$a_n = 2a_{n-1}$$

The word *homogeneous* refers to the exponents of the variables. For example $x^2 + y^2 = z^2$ is a homogeneous equation of degree 2 since each term is a square of a variable. We can add coefficients and it will still be homogeneous. Also xy has degree 2 so

$$4x^2 + 5y^2 + xy = 7z^2$$

is a homogeneous equation.

The solution of the homogeneous equation will have the following form:

$$a_n = \lambda^n$$

for some *complex number* λ. We just need to find λ. To do this we plug into the equation:

$$\lambda^n = 2\lambda^{n-1}$$

This gives $\lambda = 2$ (or $\lambda = 0$. But we only want the nonzero solutions.)

We need to check that this works. If $a_n = 2^n$ then $a_{n-1} = 2^{n-1}$ and

$$2a_{n-1} = 2 \cdot 2^{n-1} = 2^n = a_n$$

So, $a_n = 2^n$ is a solution of the homogeneous equation.

Next we need the following theorem:

Theorem 7.10. *The general solution of a first order linear recurrence is given by*

$$a_n = (\textit{particular solution}) + x(\textit{homogeneous solution})$$

In this case we get:

$$a_n = (-1) + 2^n x$$

7.4. **theory of homogeneous linear recurrences.**

Definition 7.11. A function in one variable $f(x)$ or several variables, say $f(x, y, z)$ is called **homogeneous** of degree d if, for any constant a, multiplication of all the input variables by a will multiply the value of the function by a^d:

$$f(ax) = a^d f(x), \quad f(ax, ay, az) = a^d f(x, y, z)$$

Example 7.12. The function $f(x, y) = x^2 + xy$ is homogeneous of degree 2 since

$$\begin{aligned} f(ax, ay) &= (ax)^2 + axay = a^2x^2 + a^2xy \\ &= a^2(x^2 + xy) = a^2 f(x, y) \end{aligned}$$

The function is homogeneous since both of the terms x^2 and xy have degree 2. The second term x^1y^1 has degree $1 + 1 = 2$. The degree is the sum of the exponents in the variables.

Example 7.13. The function $f(x) = 2x + 1$ is not homogeneous

$$f(ax) = 2ax + 1 \neq a^d(2x + 1)$$

(since you can't factor out the a). You can also see that this is not homogeneous since the two terms have different degrees. $2x$ has degree 1 and the constant term 1 has degree 0.

Definition 7.14. A **homogeneous linear recurrence** of order k is given by

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, c_2, \cdots, c_k$ are constants.

Example 7.15.

$$a_n = 2a_{n-1} + a_{n-2}$$

$$a_n = a_{n-1} + a_{n-2}$$

are homogeneous linear recurrences of order 2. We will solve the second one below.

Definition 7.16. A **vector space** is a nonempty set of vectors which is closed under addition and scalar multiplication.

Example 7.17. Any (infinite, straight) line through the origin or any plane through the origin is a vector space. A straight line which does not pass through the origin is not a vector space since it is not closed under scalar multiplication.

The purpose of this definition is to compress the idea into two words.

Theorem 7.18. *The set of solutions of a homogeneous linear recurrence is a vector space. In other words, given any two solutions* $\langle a\rangle, \langle b\rangle$ *and any two scalars* x, y, *the linear combination*

$$d_n = xa_n + yb_n$$

gives another solution $\langle d\rangle$ *of the same homogeneous linear recurrence.*

The first sentence has the same meaning as the second sentence.

Proof. You just expand the terms:

$$\begin{aligned} d_n := xa_n + yb_n = x(c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}) \\ +y(c_1b_{n-1} + c_2b_{n-2} + \cdots + c_kb_{n-k}) \\ = c_1(xa_{n-1} + yb_{n-1}) + c_2(xa_{n-2} + yb_{n-2}) + \cdots + c_k(xa_{n-k} + yb_{n-k}) \\ = c_1d_{n-1} + c_2d_{n-2} + \cdots + c_ka_{n-k} \end{aligned}$$

This shows that $\langle d\rangle = x\langle a\rangle + y\langle b\rangle = \langle xa_n + yb_n\rangle$ satisfies the recurrence. □

Theorem 7.19. *A homogeneous linear recurrence of order* k *has* k *basic solutions and the general solution is given by a linear combination of these* k *basic solutions.*

For example, a second order linear recurrence has two basic solutions a_n, b_n and the general solution is given by a linear combination of these.

7.5. **Fibonacci sequence.** Next, we used this theory to solve (find an equation for) the Fibonacci sequence.

(1) First we solve the homogeneous linear recurrence

$$a_n = a_{n-1} + a_{n-2}$$

(2) Then we plug in the initial conditions $a_0 = 0, a_1 = 1$ to get the specific formula for the Fibonacci sequence. (The book starts with $a_0 = 1, a_1 = 1$ which shifts the sequence one step.)

7.5.1. *formula of solving homogeneous recurrences.* There is a simple formula for solving these problems which usually works. There are exceptional cases which I will explain next week.

The basic solution of any homogeneous linear recurrence is given by

$$a_n = \lambda^n$$

where $\lambda \neq 0$. You just need to figure out what is λ.

Plugging this into the recurrence we get:

$$\lambda^n = \lambda^{n-1} + \lambda^{n-2}$$

Since $\lambda \neq 0$ we get:

$$\lambda^2 = \lambda + 1$$

or

$$\lambda^2 - \lambda - 1 = 0$$
$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

So, the two basic solutions are

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \qquad b_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Since the Fibonacci recurrence has order $k = 2$ we have enough solutions. The general solution is:

$$a_n = x\left(\frac{1+\sqrt{5}}{2}\right)^n + y\left(\frac{1-\sqrt{5}}{2}\right)^n$$

7.5.2. *using initial conditions.* There are two initial conditions: $a_0 = 0, a_1 = 1$ which are not given by the recurrence. These two conditions give two equations which we can solve for the two unknowns x, y in the general solutions.

$$a_0 = x\left(\frac{1+\sqrt{5}}{2}\right)^0 + y\left(\frac{1-\sqrt{5}}{2}\right)^0 = x + y = 0$$

or $y = -x$

$$a_1 = x\left(\frac{1+\sqrt{5}}{2}\right)^1 + y\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$
$$= x\left(\frac{1+\sqrt{5}}{2}\right)^1 - x\left(\frac{1-\sqrt{5}}{2}\right)^1 = x\sqrt{5} = 1$$

So,

$$x = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

So, the final answer is:

$$a_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

7.6. **generating functions.**

Definition 7.20. The **generating function** for a sequence of numbers $\langle a\rangle = (a_0, a_1, a_2, \cdots)$ (or other quantities which can be multiplied and added) is defined to be the function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Here are some examples. Take the sequence

$$a_n = 2^n : 1, 2, 4, 8, 16, 32, \cdots$$

The generating function is:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$A(x) = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

This is a geometric series. Each term is $2x$ times the previous term. The sum is

$$\frac{\text{first term}}{1 - \text{ratio}} = \frac{1}{1-2x}$$

Proof.

$$\begin{aligned} A(x)(1-2x) &= 1 + 2x + 4x^2 + 8x^3 + \cdots \\ &\quad -2x(1 + 2x + 4x^2 + 8x^3 + \cdots) \\ &= 1 + 2x + 4x^2 + 8x^3 + \cdots \\ &\qquad -2x - 4x^2 - 8x^3 - \cdots = 1 \end{aligned}$$

□

Here is a general problem with a constant c:

$$a_n = c^n$$

Then the generating function is

$$A(x) = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \frac{1}{1-cx}$$

Now take the derivative of each side!

$$A'(x) = c + 2c^2x + 3c^3x^2 + 4c^4x^3 + \cdots = \frac{c}{(1-cx)^2}$$

Multiply both sides by x:

$$xA'(x) = cx + 2c^2x^2 + 3c^3x^3 + 4c^4x^4 + \cdots = \frac{cx}{(1-cx)^2}$$

$$\boxed{\sum_{n=0}^{\infty} nc^n x^n = \frac{cx}{(1-cx)^2}}$$

The important part is the denominator of the fraction:

$$(1 - cx)^2 = 1 - 2cx + c^2x^2$$

This is a polynomial which has a double root.

7.6.1. *using generating functions to solve recurrence relations.* Start with

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial conditions:

$$a_0 = 0, \quad a_1 = 1$$

Multiply by x^n:

$$a_nx^n = 4a_{n-1}x^n - 4a_{n-2}x^n$$

Rewrite this:

$$a_nx^n = 4xa_{n-1}x^{n-1} - 4x^2a_{n-2}x^{n-2}$$

Sum over all $n \geq 2$ (The equation only holds for $n \geq 2$)

$$\sum_{n=2}^{\infty} a_nx^n = 4x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - 4x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}$$

$$A(x) - a_0 - a_1x = 4x(A(x) - a_0) - 4x^2A(x).$$

$$A(x) - x = 4xA(x) - 4x^2A(x).$$

So,

$$A(x) = \frac{x}{(1 - 2x)^2}$$

Compare this with what we had before.

7.6.2. *particular solutions.* Before the formal beginning of the class we did the following problem:

$$a_n = 3a_{n-1} + 5, \qquad a_0 = 1$$

The homogeneous recurrence is:

$$a_n = 3a_{n-1}$$

with solution $a_n = 3^n$ and general solution $a_n = x3^n$. The answer will be the sum of the homogeneous solution with a particular solution:

$$a_n = bn + c$$

where b, c are constants. To find them, you plug into the recurrence:

$$a_{n-1} = b(n-1) + c$$

$$a_n = 3a_{n-1} + 5$$

$$bn + c = 3b(n-1) + 3c + 5$$

which has solution $b = 0$, $c = -5/2$. ($b = 0$ unless $\lambda = 1$). If you want to do this systematically, you plug in $n = 0$ and $n = 1$ and solve for b and c. Another example is:

$$a_n = a_{n-1} + 5$$

then the particular solution is:

$$a_n = bn + c$$

$$a_{n-1} = b(n-1) + c$$

$$a_n - a_{n-1} = (bn + c) - (bn - b + c) = b = 5$$

and $c = a_0$.

7.7. **double roots.** We started the class with the problem

$$a_n = 4a_{n-1} - 4a_{n-2}$$

This is a linear, constant coefficient, homogeneous second order recurrence. The theory says it has two independent solutions. a_n, b_n and the general solution is

$$xa_n + yb_n$$

where x, y are constants determined by the initial conditions.

The first solution is $a_n = \lambda^n$ for some $\lambda \neq 0$. To find λ you plug this in:

$$\lambda^n = 4\lambda^{n-1} - 4\lambda^{n-2}$$

Since $\lambda \neq 0$ we can factor out λ^{n-2} to get

$$\lambda^2 = 4\lambda - 4$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

There is only one solution $\lambda = 2$. This is a *double root* of the polynomial. Some people write $\lambda = 2, 2$ to indicate that we get the same solution twice. The theorem is that when λ is a double root the basic solutions are

$$\lambda^n, \quad n\lambda^n$$

So, the solutions in this case are

$$2^n, \quad n2^n$$

The general solution is given by

$$a_n = x2^n + yn2^n$$

where x, y are given by the initial values a_0, a_1.

7.8. **more on generating functions.**

Theorem 7.21. *If* $\langle a \rangle = (a_0, a_1, a_2, \cdots)$ *is a sequence of numbers satisfying a homogeneous linear (constant coefficient) recurrence of order* k *then the generating function*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a rational function of x*, i.e., a fraction*

$$A(x) = \frac{f(x)}{g(x)}$$

where $f(x), g(x)$ *are polynomials of degree* $< k, = k$ *respectively.*

Proof. I gave the proof only in the case $k = 2$. To do this proof, we needed take an arbitrary homogeneous linear second order recurrence:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

with arbitrary initial conditions: $a_0, a_1 =$ fixed constants.

[One important logical step which I didn't emphasize is that we know from the previous theorem that this system of equation has a unique solution. So, we continue with the assumption that the solution exists.]

We went through the process:

$$a_n x^n = c_1 x a_{n-1} x^{n-1} + c_2 x^2 a_{n-2} x^{n-2}$$

sum over $n \geq 2$ (where the equation holds) to get:

$$\sum_{n=2}^{\infty} a_n x^n = c_1 x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + c_2 x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$A(x) - a_0 - a_1 x = c_1 x (A(x) - a_0) + c_2 x^2 A(x)$$

Now we collect $A(x)$ to one side to get:

$$A(x)(1 - c_1 x - c_2 x^2) = a_0 + a_1 x - a_0 c_1 x$$

So:

$$A(x) = \frac{a_0 + a_1 x - a_0 c_1 x}{1 - c_1 x - c_2 x^2}$$

This is a fraction with numerator $f(x) = a_0 + a_1 x - a_0 c_1 x$ which is a polynomial of degree $1 < 2$ and denominator $g(x) = 1 - c_1 x - c_2 x^2$ a polynomial of degree equal to 2. This proves the theorem (for $k = 2$). □

To actually solve the recurrence we need to factor the denominator and use partial fractions.

7.8.1. *nonhomogeneous recurrence.* We also did a nonhomogeneous recurrence using generating functions. We needed the following equations:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

Now if we started with

$$a_n = 7a_{n-1} + n + 1$$

with initial condition $a_0 = 1$ then we multiply x^n and sum over all $n \geq 1$ to get:

$$\sum_{n=1}^{\infty} a_n x^n = 7x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} nx^n + \sum_{n=1}^{\infty} x^n$$

$$A(x) - a_0 = 7xA(x) + \frac{x}{(1-x)^2} + \frac{1}{1-x} - 1$$

$$A(x) - 1 = 7xA(x) + \frac{x}{(1-x)^2} + \frac{x}{1-x}$$

$$A(x)(1-7x) = \frac{x}{(1-x)^2} + \frac{x}{1-x} + 1$$

$$A(x) = \frac{1}{(1-x)^2(1-7x)}$$

Then we need to do partial fractions:

$$A(x) = \frac{1}{(1-x)^2(1-7x)} = \frac{Ax+B}{(1-x)^2} + \frac{C}{1-7x}$$

$$A(x) = \frac{Ax}{(1-x)^2} + \frac{B}{(1-x)^2} + \frac{C}{1-7x}$$

This implies that

$$a_n = An + B(n+1) + C7^n$$

The other method was by homogeneous and particular solutions: The particular solution was obtained by

$$a_n = bn + c$$

I said b is usually zero. I meant to say that $a = 0$ and the general equation is

$$a_n = an^2 + bn + c$$

You solve for a, b, c by plugging this solution into the recursion and you get $a = 0$, $b = -1/6$, $c = -13/36$. Then add the homogeneous solution times x:

$$a_n = x7^n - \frac{1}{6}n - \frac{13}{36}$$

plug in $n = 0$, $a_0 = 1$ to get

$$1 = x - \frac{13}{36}$$

$$x = \frac{49}{36}$$

$$\boxed{a_n = \frac{49}{36}7^n - \frac{1}{6}n - \frac{13}{36}}$$

7.9. **transition to Calculus.** The last topic will be the theory behind the Calculus. But I want to use solutions of recurrences as a stepping stone.

Two topics I want to discuss are:

(1) $\frac{d}{dn}$: Differentiation with respect to n
(2) Convergence of sequences and series.

Example 7.22. Take the first order linear recurrence

$$a_n = a_{n-1} + 4n + 1$$

with initial condition $a_0 = 6$. In this method, we view n as time and we differentiate:

$$a_n' = a_{n-1}' + 4$$

$$a_n'' = a_{n-1}''$$

This means $a_n'' = C_1$ is a constant. Integrate to get

$$a_n' = \int a_n'' \, dn = C_1 n + C_2$$

Plug in $n - 1$:

$$a_{n-1}' = C_1(n-1) + C_2 = C_1 n - C_1 + C_2$$

Subtract to get:

$$a_n - a_{n-1} = C_1$$

But we know that $a_n - a_{n-1} = 4$. So, $C_1 = 4$.

$$a_n' = 4n + C_2$$

$$a_n = \int 4n + C_2 \, dn = 2n^2 + C_2 n + C_3$$

$$a_{n-1} = 2(n-1)^2 + C_2(n-1) + C_3$$

$$= 2n^2 - 4n + 2 + C_2 n - C_2 + C_3$$

Subtract from a_n to get

$$a_n - a_{n-1} = 4n - 2 + C_2 = 4n + 1$$

So, $C_2 = 3$.

$$a_n = 2n^2 + 3n + C_3$$

Plug in $n = 0$ to get $C_3 = a_0 = 6$. So the answer is:

$$a_n = 2n^2 + 3n + 6$$

Example 7.23. The coefficient 1 is special. Here is a more general case:

$$a_n = 3a_{n-1} + 8$$

with initial value $a_0 = 5$. Differentiating with respect to n gives:

$$a'_n = 3a'_{n-1}$$

This is homogeneous with solution $a'_n = C_1 3^n$.

$$a_n = \int C_1 3^n \, dn = \frac{C_1}{\ln 3} 3^n + C_2 = C_3 3^n + C_2$$

$$3a_{n-1} = 3C_3 3^{n-1} + 3C_2$$

so,

$$a_n - 3a_{n-1} = -2C_2 = 8$$

So, $C_2 = -4$

$$a_n = C_3 3^n - 4$$

Plug in $n = 0$ to get

$$a_0 = 5 = C_3 - 4$$

So, $C_3 = 9$ and the final answer is:

$$a_n = 3^{n+2} - 4$$

The theorem is that the recurrence relation was properly set up (with all necessary initial conditions). So, the solution exists and is unique. Therefore, it is legitimate to start with the assumption that we have a solution. We concluded that the solution must be $a_n = 3^{n+2} - 4$. Since the solution is unique. This must be the solution.

Example 7.24. Try this:

$$a_n = 2a_{n-1} + n + 1$$

with initial value $a_0 = 5$.

8. Limits and convergence

In this final section we will study the two basic concepts of calculus which are

(1) Infinite sums
(2) Infinitesimal differences

These are both limits: one goes to ∞ and one goes to 0. The first concept is much easier and was discovered by the ancient Greeks. The second concept was discovered by Newton but first published by Leibnitz. (but they didn't want to give the other person any credit.)

Logically, the first principle is the one which came historically last, namely the concept of convergence of a sequence. But I will start with infinite series because history shows that this is the easiest for people to understand.

8.1. **infinite sums.**

In the study of recurrence relations we saw geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

If $|r| < 1$ then this sum is finite

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

What I want to discuss is the *definition* of the infinite sum. What does this equation mean?

8.1.1. *least upper bounds.* The first definition is in terms of one of the basic axiomatic properties of real numbers.

Axiom 8.1 (Completeness Axiom for real numbers)**.** *Every nonempty subset of real numbers with an upper bound has a least upper bound.*

Remember that an **upper bound** for a subset $S \subseteq \mathbb{R}$ is a real number α so that $x \leq \alpha$ for all $x \in S$. The **least upper bound** or **supremum** of S is defined to be the smallest upper bound:

$$\sup S = \beta$$

means β is an upper bound for S and $\beta \leq \alpha$ for any other upper bound for S. The Completeness Axiom says that such a real number β exists.

Suppose that $a_0, a_1, a_2, \cdots$ is a sequence of nonnegative real numbers $a_n \geq 0$. The, by a **partial sum** of these numbers we mean the sum of the first $n+1$ terms:

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n$$

This is a nondecreasing sequence:

$$s_0 \leq s_1 \leq s_2 \leq \cdots$$

Proposition 8.2. *This sequence is either bounded or unbounded.*

In our case, we need upper bounds. There are two cases:

(1) The set of numbers s_n has an upper bound.
(2) It doesn't have an upper bound.

This is obviously a true statement and we will use symbolic logic to figure out what these mean.

Case 1:

$$(\exists M \in \mathbb{R}) \underbrace{(\forall n \in \mathbb{Z}_{\geq 0}) \quad s_n \leq M}_{M \text{ is an upper bound for } \langle s \rangle}$$

We had a discussion about the rephrasing of the statement that M is an upper bound for the set of all partial sums. Another way to write this is:

$$(\forall x \in \{s_n\}) \quad x \leq M$$

However, this has a variable set inside the quantifier. This is legal but undesirable.

Quantifiers should use *fixed* sets.

Case 2: The negation of this statement is:

$$(\forall M \in \mathbb{R})(\exists n \in \mathbb{Z}_{\geq 0}) \quad s_n > M$$

In other words, the numbers s_n become arbitrarily large (larger than any number M that you can give me.

Interpretation: The sequence of numbers s_n is going to $+\infty$.

Definition 8.3. If $a_n \geq 0$ then the infinite sum $\sum_{n=0}^{\infty}$ is defined as follows. In Case 2, when the set of partial sums is unbounded, we define the infinite sum to be $+\infty$. In Case 1, when the set of partial sums s_n has an upper bound, we define the sum to be the supremum:

$$\sum_{n=0}^{\infty} a_n := \sup(s_n : n \geq 0)$$

In order for this definition to make sense we need another theorem:

Theorem 8.4. *If a nonempty subset S of $\mathbb{R}$ has an upper bound then it has a unique least upper bound.*

8.2. **limits.** The completeness axiom tells us that every bounded set has a least upper bound. This allows us to "construct" the limit of an increasing sequence. Next, we look at any sequence and use the "squeeze theorem" But first we need the *definition of a limit* so that we know exactly what we are computing.

Definition 8.5. The **limit** of a sequence of real numbers $\langle a\rangle = (a_0, a_1, a_2, \cdots)$ is defined to be a real number L with the following property where P is the set of positive real numbers.

$$(\forall \epsilon \in P)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n > N \Rightarrow |a_n - L| < \epsilon]$$

We write:

$$\lim_{n\to\infty} a_n = L$$

and we say that the sequence a_n *converges* to L.

This definition is almost always written in the following sloppy form:

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)|a_n - L| < \epsilon$$

When you translate this into words, you need to insert the missing assumptions:

For all positive real numbers ϵ there is a positive integer N so that for all integers $n > N$, $|a_n - L| < \epsilon$.

This definition says that the sequence a_n gets "arbitrarily close" to L as n goes to ∞.

Example 8.6. Show that the sequence $a_n = 1/n$ converges to 0.

The statement that we need to prove is:

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)|1/n| < \epsilon$$

The proof is: For any $\epsilon > 0$ let N be an integer greater than $1/\epsilon$. Then for any $n > N$ we have

$$n > N > 1/\epsilon > 0$$

so

$$0 < \frac{1}{n} < \frac{1}{N} < \epsilon$$

proving the statement. □