6.9. **Problems.** (1) Suppose that G is a bipartite graph with parts A, B. If every vertex has degree 3 prove that $|A| = |B|$.

[Find a combinatorial proof: Count the number of edges in G in two different ways.]

(2) [Similar to HW7, 11.19(a)] Find a combinatorial proof of the formula:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

by counting the number of subsets of $[n+1]$ with k elements in two different ways. [Assume the formula given in class without a complete proof: $\binom{n}{k}$ is the number of k element subsets of a set with n elements.]

(3) A graph has the property that every vertex has degree 5. Show that there are an even number of vertices. [Assume that the number of vertices is odd and get a contradiction.]

6.10. **theorems.**

Lemma 6.7. *Every tree is bipartite and there are exactly two ways to color it with two colors.*

Theorem 6.8. *A graph Γ is bipartite if and only if it has no odd cycles.*

Proof. If Γ is bipartite, it is 2 colorable. So it cannot have an odd cycle since, the colors on vertices in a cycle must alternate. Conversely, suppose that Γ has no odd cycles. Then, each component of Γ has a spanning tree which is 2-colorable. Each additional edge (called a **chord**) must connect two vertices of oppose colors since each chord forms an even cycle with the tree. So, the coloring of the spanning trees gives a coloring of the graph. □

Lemma 6.9. *The sum of the degrees of the vertices in a graph is always even.*

Theorem 6.10. *A graph Γ is Eulerian if and only if it is connected and either*

(1) *Γ has exactly two vertices of odd degree.*
(2) *all vertices of Γ have even degree*

In the first case, there is an Eulerian trail connecting the two vertices of odd degree, in the second case there is a closed Eulerian trail starting at any point.

Here is Pascal's triangle which illustrates problem (2):

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Each entry is the sum of the two above it:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

So, for example:

$$
\begin{array}{ccccc}
\begin{pmatrix} 5 \\ 3 \end{pmatrix} & = & \begin{pmatrix} 4 \\ 3 \end{pmatrix} & + & \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\
10 & = & 6 & + & 4
\end{array}
$$