7. Recurrence relations

7.1. **basic definitions.** Notation: By a **sequence** we mean an infinite sequence:

$$a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \cdots$$

The sequence is denoted $\langle a \rangle$. The a_i are elements of some set S. And strictly speaking the sequence is a function

$$a : \mathbb{Z}_{\geq 0} \to S$$

For example, $a_n = 2n + 1$ is a formula for the sequence of integers

$$\langle a \rangle = (1, 3, 5, 7, 9, 11, \cdots)$$

Definition 7.1. A **recurrence relation** of **order** k is an expression or formula which, for $n \geq k$ gives a_n in terms of the quantities n and the previous k terms of the sequence which are:

$$a_{n-k}, \cdots, a_{n-2}, a_{n-1}$$

Example 7.2. For example, the famous Fibonacci sequence:

$$\langle a \rangle = (0, 1, 1, 2, 3, 5, 8, 13, \cdots)$$

is given by the second order recurrence:

$$a_n = a_{n-1} + a_{n-2}$$

Since $k = 2$, this formula only holds for $n \geq 2$.

$$a_0 = 1, a_1 = 1$$

are not given by the recurrence relation.

Example 7.3. The odd number sequence $a_n = 2n + 1$ can be given by a recurrence of order 1:

$$a_n = a_{n-1} + 2$$

starting with $a_0 = 1$. But the same recurrence relation gives the even numbers if we start with $a_0 = 2$. So, in this example, we have two sequences satisfying the same recurrence.

Definition 7.4. If a sequence $\langle a \rangle = (a_0, a_1, \cdots)$ satisfies a recurrence relation of order k, the values of $a_0, a_1, \cdots, a_{k-1}$ are called the **initial values** of the sequence.

Theorem 7.5. *Given any recurrence relation of order k, the initial values of a sequence can be chosen arbitrarily and the rest of the sequence will be uniquely determined.*

7.2. **problems.**

Example 7.6. Draw 9 straight lines which are not parallel (so every pair of lines meets at a point) and so that no three lines go through the same point. The lines divide the plane into regions. How many regions do you get? Write down a recurrence relation for the number of regions you get with n lines.

The way to solve problems like this is to first do examples with small numbers. Take $n = 0, 1, 2, 3, 4, 5$ and count the number of regions. (With 0 lines you get $a_0 = 1$ region, with 1 line you get $a_1 = 2$ regions, etc.) You should see a pattern, especially for the difference between the numbers.

Calculations gave:

$$\begin{array}{cccccc} a_0, & a_1, & a_2, & a_3, & a_4, & a_5 \\ 1, & 2, & 4, & 7, & 11, & 16, \end{array}$$

with differences:

$$1, \quad 2, \quad 3, \quad 4, \quad 5,$$

So, the recurrence is

$$a_n = a_{n-1} + n$$

This has order $k = 1$. The proof that this holds is: The n-th line meets each of the previous $n-1$ lines cutting each of the regions bounded by those line into two regions. The number of regions is n since the strip around the n-th line is cut $n-1$ times by these lines.

Example 7.7. Find all possible real number solutions to the first order recurrence:

$$a_n = a_{n-1} + 2$$

Here $k = 1$. So, the first value a_0 can be chosen arbitrarily. So, set it equal to $a_0 = x$. Next, what is a_1? a_2? a_n? Congratulations! You solved the problem.

This was easy:

$$a_n = x + 2$$

where x is any real number.

Example 7.8. Write down a recurrence relation for the numbers $a_n = n!$. What are the initial values? What is the order of the recurrence?

The recurrence is $a_n = n \cdot a_{n-1}$ this has order $k = 1$ with initial value $a_0 = 1$. (This is given by the definition $0! = 1$.)

7.3. **solution of linear 1st order recurrence.** I also had time to explain the solution of a typical first order linear recurrence.

Example 7.9. Find the general solution of the recursion:

$$a_n = 2a_{n-1} + 1$$

Since $k = 1$, we can start with any value of a_0 and the rest of the sequence will be determined. One amusing value is

$$a_0 = -1$$
$$a_1 = 2a_0 + 1 = 2(-1) + 1 = -1$$
$$a_2 = -1, a_3 = -1, \cdots, a_n = -1$$

This is called a **particular solution** to the recurrence. The theory, which I will explain later says we should combine this with a solution of:

The **associated homogeneous recurrence** which is:

$$a_n = 2a_{n-1}$$

The word *homogeneous* refers to the exponents of the variables. For example $x^2 + y^2 = z^2$ is a homogeneous equation of degree 2 since each term is a square of a variable. We can add coefficients and it will still be homogeneous. Also xy has degree 2 so

$$4x^2 + 5y^2 + xy = 7z^2$$

is a homogeneous equation.

The solution of the homogeneous equation will have the following form:

$$a_n = \lambda^n$$

for some *complex number* λ. We just need to find λ. To do this we plug into the equation:

$$\lambda^n = 2\lambda^{n-1}$$

This gives $\lambda = 2$ (or $\lambda = 0$. But we only want the nonzero solutions.)

We need to check that this works. If $a_n = 2^n$ then $a_{n-1} = 2^{n-1}$ and

$$2a_{n-1} = 2 \cdot 2^{n-1} = 2^n = a_n$$

So, $a_n = 2^n$ is a solution of the homogeneous equation.

Next we need the following theorem:

Theorem 7.10. *The general solution of a first order linear recurrence is given by*

$$a_n = (\textit{particular solution}) + x(\textit{homogeneous solution})$$

In this case we get:

$$a_n = (-1) + 2^n x$$

7.4. **theory of homogeneous linear recurrences.**

Definition 7.11. A function in one variable $f(x)$ or several variables, say $f(x, y, z)$ is called **homogeneous** of degree d if, for any constant a, multiplication of all the input variables by a will multiply the value of the function by a^d:

$$f(ax) = a^d f(x), \quad f(ax, ay, az) = a^d f(x, y, z)$$

Example 7.12. The function $f(x, y) = x^2 + xy$ is homogeneous of degree 2 since

$$\begin{aligned} f(ax, ay) = (ax)^2 + axay &= a^2x^2 + a^2xy \\ &= a^2(x^2 + xy) = a^2 f(x, y) \end{aligned}$$

The function is homogeneous since both of the terms x^2 and xy have degree 2. The second term x^1y^1 has degree $1 + 1 = 2$. The degree is the sum of the exponents in the variables.

Example 7.13. The function $f(x) = 2x + 1$ is not homogeneous

$$f(ax) = 2ax + 1 \neq a^d(2x + 1)$$

(since you can't factor out the a). You can also see that this is not homogeneous since the two terms have different degrees. $2x$ has degree 1 and the constant term 1 has degree 0.

Definition 7.14. A **homogeneous linear recurrence** of order k is given by

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, c_2, \cdots, c_k$ are constants.

Example 7.15.

$$a_n = 2a_{n-1} + a_{n-2}$$
$$a_n = a_{n-1} + a_{n-2}$$

are homogeneous linear recurrences of order 2. We will solve the second one below.

Definition 7.16. A **vector space** is a nonempty set of vectors which is closed under addition and scalar multiplication.

Example 7.17. Any (infinite, straight) line through the origin or any plane through the origin is a vector space. A straight line which does not pass through the origin is not a vector space since it is not closed under scalar multiplication.

The purpose of this definition is to compress the idea into two words.

Theorem 7.18. *The set of solutions of a homogeneous linear recurrence is a vector space. In other words, given any two solutions $\langle a\rangle, \langle b\rangle$ and any two scalars x, y, the linear combination*

$$d_n = xa_n + yb_n$$

gives another solution $\langle d\rangle$ of the same homogeneous linear recurrence.

The first sentence has the same meaning as the second sentence.

Proof. You just expand the terms:

$$\begin{aligned} d_n := xa_n + yb_n = x(c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}) \\ +y(c_1b_{n-1} + c_2b_{n-2} + \cdots + c_kb_{n-k}) \\ = c_1(xa_{n-1} + yb_{n-1}) + c_2(xa_{n-2} + yb_{n-2}) + \cdots + c_k(xa_{n-k} + yb_{n-k}) \\ = c_1d_{n-1} + c_2d_{n-2} + \cdots + c_ka_{n-k} \end{aligned}$$

This shows that $\langle d\rangle = x\langle a\rangle + y\langle b\rangle = \langle xa_n + yb_n\rangle$ satisfies the recurrence. □

Theorem 7.19. *A homogeneous linear recurrence of order k has k basic solutions and the general solution is given by a linear combination of these k basic solutions.*

For example, a second order linear recurrence has two basic solutions a_n, b_n and the general solution is given by a linear combination of these.

7.5. **Fibonacci sequence.** Next, we used this theory to solve (find an equation for) the Fibonacci sequence.

(1) First we solve the homogeneous linear recurrence

$$a_n = a_{n-1} + a_{n-2}$$

(2) Then we plug in the initial conditions $a_0 = 0, a_1 = 1$ to get the specific formula for the Fibonacci sequence. (The book starts with $a_0 = 1, a_1 = 1$ which shifts the sequence one step.)

7.5.1. *formula of solving homogeneous recurrences.* There is a simple formula for solving these problems which usually works. There are exceptional cases which I will explain next week.

The basic solution of any homogeneous linear recurrence is given by

$$a_n = \lambda^n$$

where $\lambda \neq 0$. You just need to figure out what is λ.

Plugging this into the recurrence we get:

$$\lambda^n = \lambda^{n-1} + \lambda^{n-2}$$

Since $\lambda \neq 0$ we get:

$$\lambda^2 = \lambda + 1$$

or

$$\lambda^2 - \lambda - 1 = 0$$
$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

So, the two basic solutions are

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \qquad b_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Since the Fibonacci recurrence has order $k = 2$ we have enough solutions. The general solution is:

$$a_n = x\left(\frac{1+\sqrt{5}}{2}\right)^n + y\left(\frac{1-\sqrt{5}}{2}\right)^n$$

7.5.2. *using initial conditions.* There are two initial conditions: $a_0 = 0, a_1 = 1$ which are not given by the recurrence. These two conditions give two equations which we can solve for the two unknowns x, y in the general solutions.

$$a_0 = x\left(\frac{1+\sqrt{5}}{2}\right)^0 + y\left(\frac{1-\sqrt{5}}{2}\right)^0 = x + y = 0$$

or $y = -x$

$$a_1 = x\left(\frac{1+\sqrt{5}}{2}\right)^1 + y\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$
$$= x\left(\frac{1+\sqrt{5}}{2}\right)^1 - x\left(\frac{1-\sqrt{5}}{2}\right)^1 = x\sqrt{5} = 1$$

So,

$$x = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

So, the final answer is:

$$a_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

7.6. generating functions.

Definition 7.20. The **generating function** for a sequence of numbers $\langle a\rangle = (a_0, a_1, a_2, \cdots)$ (or other quantities which can be multiplied and added) is defined to be the function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Here are some examples. Take the sequence

$$a_n = 2^n : 1, 2, 4, 8, 16, 32, \cdots$$

The generating function is:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$A(x) = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

This is a geometric series. Each term is $2x$ times the previous term. The sum is

$$\frac{\text{first term}}{1 - \text{ratio}} = \frac{1}{1-2x}$$

Proof.

$$\begin{aligned} A(x)(1-2x) &= 1 + 2x + 4x^2 + 8x^3 + \cdots \\ &\quad -2x(1 + 2x + 4x^2 + 8x^3 + \cdots) \\ &= 1 + 2x + 4x^2 + 8x^3 + \cdots \\ &\qquad -2x - 4x^2 - 8x^3 - \cdots = 1 \end{aligned}$$

□

Here is a general problem with a constant c:

$$a_n = c^n$$

Then the generating function is

$$A(x) = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \frac{1}{1-cx}$$

Now take the derivative of each side!

$$A'(x) = c + 2c^2x + 3c^3x^2 + 4c^4x^3 + \cdots = \frac{c}{(1-cx)^2}$$

Multiply both sides by x:

$$xA'(x) = cx + 2c^2x^2 + 3c^3x^3 + 4c^4x^4 + \cdots = \frac{cx}{(1-cx)^2}$$

$$\boxed{\sum_{n=0}^{\infty} nc^n x^n = \frac{cx}{(1-cx)^2}}$$

The important part is the denominator of the fraction:

$$(1 - cx)^2 = 1 - 2cx + c^2x^2$$

This is a polynomial which has a double root.

7.6.1. *using generating functions to solve recurrence relations.* Start with

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial conditions:

$$a_0 = 0, \quad a_1 = 1$$

Multiply by x^n:

$$a_nx^n = 4a_{n-1}x^n - 4a_{n-2}x^n$$

Rewrite this:

$$a_nx^n = 4xa_{n-1}x^{n-1} - 4x^2a_{n-2}x^{n-2}$$

Sum over all $n \geq 2$ (The equation only holds for $n \geq 2$)

$$\sum_{n=2}^{\infty} a_nx^n = 4x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - 4x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}$$

$$A(x) - a_0 - a_1x = 4x(A(x) - a_0) - 4x^2A(x).$$

$$A(x) - x = 4xA(x) - 4x^2A(x).$$

So,

$$A(x) = \frac{x}{(1-2x)^2}$$

Compare this with what we had before.

7.6.2. *particular solutions.* Before the formal beginning of the class we did the following problem:

$$a_n = 3a_{n-1} + 5, \qquad a_0 = 1$$

The homogeneous recurrence is:

$$a_n = 3a_{n-1}$$

with solution $a_n = 3^n$ and general solution $a_n = x3^n$. The answer will be the sum of the homogeneous solution with a particular solution:

$$a_n = bn + c$$

where b, c are constants. To find them, you plug into the recurrence:

$$a_{n-1} = b(n-1) + c$$

$$a_n = 3a_{n-1} + 5$$

$$bn + c = 3b(n-1) + 3c + 5$$

which has solution $b = 0$, $c = -5/2$. ($b = 0$ unless $\lambda = 1$). If you want to do this systematically, you plug in $n = 0$ and $n = 1$ and solve for b and c. Another example is:

$$a_n = a_{n-1} + 5$$

then the particular solution is:

$$a_n = bn + c$$

$$a_{n-1} = b(n-1) + c$$

$$a_n - a_{n-1} = (bn + c) - (bn - b + c) = b = 5$$

and $c = a_0$.

7.7. **double roots.** We started the class with the problem

$$a_n = 4a_{n-1} - 4a_{n-2}$$

This is a linear, constant coefficient, homogeneous second order recurrence. The theory says it has two independent solutions. a_n, b_n and the general solution is

$$xa_n + yb_n$$

where x, y are constants determined by the initial conditions.

The first solution is $a_n = \lambda^n$ for some $\lambda \neq 0$. To find λ you plug this in:

$$\lambda^n = 4\lambda^{n-1} - 4\lambda^{n-2}$$

Since $\lambda \neq 0$ we can factor out λ^{n-2} to get

$$\lambda^2 = 4\lambda - 4$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

There is only one solution $\lambda = 2$. This is a *double root* of the polynomial. Some people write $\lambda = 2, 2$ to indicate that we get the same solution twice. The theorem is that when λ is a double root the basic solutions are

$$\lambda^n, \quad n\lambda^n$$

So, the solutions in this case are

$$2^n, \quad n2^n$$

The general solution is given by

$$a_n = x2^n + yn2^n$$

where x, y are given by the initial values a_0, a_1.

7.8. more on generating functions.

Theorem 7.21. *If $\langle a\rangle = (a_0, a_1, a_2, \cdots)$ is a sequence of numbers satisfying a homogeneous linear (constant coefficient) recurrence of order k then the generating function*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a rational function of x, i.e., a fraction

$$A(x) = \frac{f(x)}{g(x)}$$

where $f(x), g(x)$ are polynomials of degree $< k, = k$ respectively.

Proof. I gave the proof only in the case $k = 2$. To do this proof, we needed take an arbitrary homogeneous linear second order recurrence:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

with arbitrary initial conditions: $a_0, a_1 =$ fixed constants.

[One important logical step which I didn't emphasize is that we know from the previous theorem that this system of equation has a unique solution. So, we continue with the assumption that the solution exists.]

We went through the process:

$$a_n x^n = c_1 x a_{n-1} x^{n-1} + c_2 x^2 a_{n-2} x^{n-2}$$

sum over $n \geq 2$ (where the equation holds) to get:

$$\sum_{n=2}^{\infty} a_n x^n = c_1 x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + c_2 x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$A(x) - a_0 - a_1 x = c_1 x(A(x) - a_0) + c_2 x^2 A(x)$$

Now we collect $A(x)$ to one side to get:

$$A(x)(1 - c_1 x - c_2 x^2) = a_0 + a_1 x - a_0 c_1 x$$

So:

$$A(x) = \frac{a_0 + a_1 x - a_0 c_1 x}{1 - c_1 x - c_2 x^2}$$

This is a fraction with numerator $f(x) = a_0 + a_1 x - a_0 c_1 x$ which is a polynomial of degree $1 < 2$ and denominator $g(x) = 1 - c_1 x - c_2 x^2$ a polynomial of degree equal to 2. This proves the theorem (for $k = 2$). □

To actually solve the recurrence we need to factor the denominator and use partial fractions.

7.8.1. *nonhomogeneous recurrence.* We also did a nonhomogeneous recurrence using generating functions. We needed the following equations:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

Now if we started with

$$a_n = 7a_{n-1} + n + 1$$

with initial condition $a_0 = 1$ then we multiply x^n and sum over all $n \geq 1$ to get:

$$\sum_{n=1}^{\infty} a_n x^n = 7x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} nx^n + \sum_{n=1}^{\infty} x^n$$

$$A(x) - a_0 = 7xA(x) + \frac{x}{(1-x)^2} + \frac{1}{1-x} - 1$$

$$A(x) - 1 = 7xA(x) + \frac{x}{(1-x)^2} + \frac{x}{1-x}$$

$$A(x)(1-7x) = \frac{x}{(1-x)^2} + \frac{x}{1-x} + 1$$

$$A(x) = \frac{1}{(1-x)^2(1-7x)}$$

Then we need to do partial fractions:

$$A(x) = \frac{1}{(1-x)^2(1-7x)} = \frac{Ax+B}{(1-x)^2} + \frac{C}{1-7x}$$

$$A(x) = \frac{Ax}{(1-x)^2} + \frac{B}{(1-x)^2} + \frac{C}{1-7x}$$

This implies that

$$a_n = An + B(n+1) + C7^n$$

The other method was by homogeneous and particular solutions: The particular solution was obtained by

$$a_n = bn + c$$

I said b is usually zero. I meant to say that $a = 0$ and the general equation is

$$a_n = an^2 + bn + c$$

You solve for a, b, c by plugging this solution into the recursion and you get $a = 0$, $b = -1/6$, $c = -13/36$. Then add the homogeneous solution times x:

$$a_n = x7^n - \frac{1}{6}n - \frac{13}{36}$$

plug in $n = 0$, $a_0 = 1$ to get

$$1 = x - \frac{13}{36}$$

$$x = \frac{49}{36}$$

$$\boxed{a_n = \frac{49}{36}7^n - \frac{1}{6}n - \frac{13}{36}}$$

7.9. **transition to Calculus.** The last topic will be the theory behind the Calculus. But I want to use solutions of recurrences as a stepping stone.

Two topics I want to discuss are:

(1) $\frac{d}{dn}$: Differentiation with respect to n
(2) Convergence of sequences and series.

Example 7.22. Take the first order linear recurrence

$$a_n = a_{n-1} + 4n + 1$$

with initial condition $a_0 = 6$. In this method, we view n as time and we differentiate:

$$a'_n = a'_{n-1} + 4$$

$$a''_n = a''_{n-1}$$

This means $a''_n = C_1$ is a constant. Integrate to get

$$a'_n = \int a''_n \, dn = C_1 n + C_2$$

Plug in $n - 1$:

$$a'_{n-1} = C_1(n-1) + C_2 = C_1 n - C_1 + C_2$$

Subtract to get:

$$a_n - a_{n-1} = C_1$$

But we know that $a_n - a_{n-1} = 4$. So, $C_1 = 4$.

$$a'_n = 4n + C_2$$

$$a_n = \int 4n + C_2 \, dn = 2n^2 + C_2 n + C_3$$

$$a_{n-1} = 2(n-1)^2 + C_2(n-1) + C_3$$

$$= 2n^2 - 4n + 2 + C_2 n - C_2 + C_3$$

Subtract from a_n to get

$$a_n - a_{n-1} = 4n - 2 + C_2 = 4n + 1$$

So, $C_2 = 3$.

$$a_n = 2n^2 + 3n + C_3$$

Plug in $n = 0$ to get $C_3 = a_0 = 6$. So the answer is:

$$a_n = 2n^2 + 3n + 6$$

Example 7.23. The coefficient 1 is special. Here is a more general case:

$$a_n = 3a_{n-1} + 8$$

with initial value $a_0 = 5$. Differentiating with respect to n gives:

$$a'_n = 3a'_{n-1}$$

This is homogeneous with solution $a'_n = C_1 3^n$.

$$a_n = \int C_1 3^n \, dn = \frac{C_1}{\ln 3} 3^n + C_2 = C_3 3^n + C_2$$

$$3a_{n-1} = 3C_3 3^{n-1} + 3C_2$$

so,

$$a_n - 3a_{n-1} = -2C_2 = 8$$

So, $C_2 = -4$

$$a_n = C_3 3^n - 4$$

Plug in $n = 0$ to get

$$a_0 = 5 = C_3 - 4$$

So, $C_3 = 9$ and the final answer is:

$$a_n = 3^{n+2} - 4$$

The theorem is that the recurrence relation was properly set up (with all necessary initial conditions). So, the solution exists and is unique. Therefore, it is legitimate to start with the assumption that we have a solution. We concluded that the solution must be $a_n = 3^{n+2} - 4$. Since the solution is unique. This must be the solution.

Example 7.24. Try this:

$$a_n = 2a_{n-1} + n + 1$$

with initial value $a_0 = 5$.