

4.8. **Worksheet.**

4.8.1. Draw a truth table for the statement  $(p \vee \neg q) \wedge (p \Rightarrow r)$ . Is this a tautology?

If  $A, B, C$  are sets and  $P(x), Q(x), R(x)$  are the assertions  $x \in A, x \in B, x \in C$  respectively, then write the set

$$D = \{x \in U \mid (P(x) \vee \neg Q(x)) \wedge (P(x) \Rightarrow R(x))\}$$

in terms of union, intersection, complements, etc using  $A, B, C$ . [Hint: Look at your truth table.]

Here is an answer suggested by one student:

First, convert to  $\wedge, \vee, \neg$ :

$$D = \{x \in U \mid (P(x) \vee \neg Q(x)) \wedge (\neg P(x) \vee R(x))\}$$

which becomes:

$$D = (A \cup B^c) \cap (A^c \cup C)$$

4.8.2. Is the following function bounded? increasing 1-1?

$$f : (-\infty, 0] \cup [1, \infty) \rightarrow \mathbb{R}$$

given by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \geq 1 \\ 2 - 3x & \text{otherwise} \end{cases}$$

This function is not bounded since it goes to  $\infty$  as  $x$  goes to  $-\infty$ . It is not increasing since, e.g.,  $f(0) = 2 > f(1) = 1$ . This function is 1-1 because it is decreasing. (To prove that  $f$  is a decreasing function take any  $x < y$  in the domain. Then there are three possibilities: either  $x < y \leq 0$ ,  $1 \leq x < y$  or  $x \leq 0$  and  $y \geq 1$ . In the first case,  $-3x > -3y$  since multiplying by a negative number reverses the direction of the inequality, so  $2 - 3x > 2 - 3y$ . In the second case,  $y > x > 0$  implies that  $1/x > 1/y$ . Finally, when  $x \leq 0, y \geq 1$  we have  $f(x) \geq 2$  and  $f(y) \leq 1$ . Therefore, in all three cases  $f(x) > f(y)$  making  $f$  a decreasing function.)

Find a real valued function on the half open interval  $(0, 1]$  which is not bounded, not increasing and not 1-1.

It easy to find functions with just one or two of these properties and we can just paste them together. For example:

$$f(x) = \begin{cases} 1/x & \text{if } 0 < x \leq 1/2 \\ 3 & \text{if } 1/2 < x \leq 1 \end{cases}$$

The function  $1/x$  is unbounded and not increasing on the domain  $(0, 1]$ . But it is 1-1. So we needed another function. The function  $f(x) = 3$  (a horizontal line of slope zero) is not 1-1 since it sends everything to one number (3).

4.8.3. Find for which positive integers  $n$  the following inequality holds and prove it by induction.

$$n^2 > 5n - 4$$

I claim that this is true for all  $n \geq 5$ . First, take  $n = 5$ . Then  $5^2 = 25$  which is greater than  $5 \cdot 5 - 4 = 21$ . So, the inequality holds in the base case  $n = 5$ . Now suppose by induction that the inequality holds for  $n$  and  $n \geq 5$ . Then we need to show that  $(n + 1)^2 > 5(n + 1) - 4$ . But:

$$(n + 1)^2 = n^2 + 2n + 1$$

By induction,  $n^2 > 5n - 4$ . So,

$$n^2 + 2n + 1 > (5n - 4) + 2n + 1 = 7n - 3 = [5(n + 1) - 4] + 2n - 2$$

But, we also know that  $n \geq 5$ . So,  $2n - 2 \geq 8$  making the last expression greater than  $5(n + 1) - 4$ . So

$$(n + 1)^2 > [5(n + 1) - 4] + 2n - 2 > 5(n + 1) - 4$$

which is what we wanted to show. So, the inequality holds for  $n + 1$  if it holds for  $n$ . So, by induction, it holds for all  $n \geq 5$ .

4.8.4. Write the following statement in logical notation, find its negation and write the negation without using words of negation such as “not” or “no” or “without” or “nonnegative” or “nonpositive” or any symbol with a line through it ( $\neq$ ,  $\not\leq$ , etc)

*The image of the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2$  is contained in the set of positive real numbers.*

This statement says that one set is contained in another set. If we negate this we would have the statement that the first set is not contained in the second set. (which says there exists an element of the first set which is not in the second).

In class I first rephrased the statement so that it is an implication. (The containment of one set in another is always an implication:  $x \in A \Rightarrow x \in B$ .) The statement rephrased was:

$$(\forall x \in \mathbb{R})x^2 > 0$$

The negation is

$$(\exists x \in \mathbb{R})x^2 \leq 0$$

In words this is:

*There is a real number whose square is less than or equal to zero.*

4.8.5. If  $a, b$  are positive real numbers so that  $2a - b \leq 9$  then prove that either  $a \leq 12$  or  $b > 15$ . Write your answer in complete sentences.

Suppose by contradiction that the conclusion does not hold. Then  $a > 12$  and  $b \leq 15$ . This implies that  $2a > 24$  and  $-b \leq -15$ . Adding these together we get

$$2a - b > 24 - 15 = 9$$

which contradicts the assumption that  $2a - b \leq 9$ . Therefore, the conclusion must be true and  $2a - b \leq 9$  implies that  $a \leq 12$  or  $b > 15$ .

One student said that he found a shorter proof. This proof was also intended to illustrate the concept of the *contrapositive* which is an important concept that I did not cover. The contrapositive of the statement  $p \Rightarrow q$  is the statement  $\neg q \Rightarrow \neg p$ . This is equivalent to the original statement. What I proved was the contrapositive of the theorem since I assumed that the conclusion was not true and I deduced that the assumption is not true. (So, using the contrapositive is the same as proof by contradiction.) Students often confuse this with the *converse* which is the statement  $q \Rightarrow p$  which is not equivalent to  $p \Rightarrow q$ , nor is it the negation of  $p \Rightarrow q$ . (But if you want to sound really smart, at the end of a math lecture you can ask the question: “Is the converse of your theorem true?”)