16 Notes on Jordan-Hölder

Definition 16.1. A *normal series* of a group G is a sequence of subgroups:

$$G = H_0 \geq H_1 \geq H_2 \geq \cdots \geq H_m = 1$$

so that each subgroup is normal in the previous one ($H_i \trianglelefteq H_{i-1}$). The quotient groups H_{i-1}/H_i is called a *subquotients*. A *refinement* of a normal series is another normal series obtained by adding extra terms.

Example 16.2. Here are some examples of normal series.

1. $S_n > A_n > 1$ is a normal series for $G = S_n$ for any $n \geq 1$. The associated subquotients are $S_n/A_n \cong \mathbb{Z}/2$ and $A_n/1 = A_n$.

2. For $n = 4$ this normal series has a refinement:

$$S_4 > A_4 > K > 1$$

   The subquotients are

   (a) $S_4/A_4 \cong \mathbb{Z}/2$
   (b) $A_4/K \cong \mathbb{Z}_3$ (Any group of order 3 is cyclic.)
   (c) $K \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

3. A normal series for the dihedral group $D_8 = \langle a, b | a^2, b^4, abab\rangle$ is given by

$$D_8 > \langle b \rangle > \left\langle b^2 \right\rangle > 1.$$

   All three subquotients are cyclic of order 2.

4. $G = GL_n(\mathbb{R})$ has a normal series

$$GL_n(\mathbb{R}) \geq SL_n(\mathbb{R}) \geq Z(GL_n(\mathbb{R})) \geq 1$$

   The subquotients are:

   (a) $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^\times$
   (b) $SL_n(\mathbb{R})/Z(GL_n(\mathbb{R})) = PSL_n(\mathbb{R})$ by definition of the *projective unimodular group* $PSL_n(\mathbb{R})$.
   (c) $Z(GL_n(\mathbb{R})) \cong \mathbb{R}^\times$ since only scalar multiples of the identity are central.

5. (A stupid example) Any group G has this normal series:

$$G \rhd 1.$$

   If G is a simple group, there is the only normal series for G.

Definition 16.3. A *composition series* is a normal series

$$G = G_0 > G_1 > \cdots > G_n = 1$$

so that G_i is a maximal proper normal subgroup of G_{i-1} for $1 \le i \le n$. The subquotients G_{i-1}/G_i are simple groups called the *composition factors* of G.

In Example 16.2, (1), for $n \ge 5$ and (3) are composition series but (2) is not because K is not simple. A composition series for S_4 is given by refining (3) by adding one more term: $H =$ the cyclic group generated by (12)(34):

$$S_4 > A_4 > K > H > 1$$

The composition factors are $\mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/2, \mathbb{Z}/2$. Example (4) $GL_n(\mathbb{R})$ does not have a composition series because it contains a normal subgroup isomorphic to $\mathbb{Z}$ and $\mathbb{Z}$ has an infinite "composition series":

$$\mathbb{Z} \ge p\mathbb{Z} \ge p^2\mathbb{Z} \ge p^3\mathbb{Z} \ge \cdots$$

for any prime p. (Making the subquotients $\mathbb{Z}/p\mathbb{Z}$ not uniquely determined by $\mathbb{Z}$.)

Theorem 16.4. *Every finite group has a composition series.*

Proof. Take the longest possible normal series. Then the subquotients are all simple. If one subquotient H_i/H_{i+1} is not simple then it has a nontrivial normal subgroup which, by the correspondence theorem, gives a group K between H_i and H_{i+1} extending the normal series:

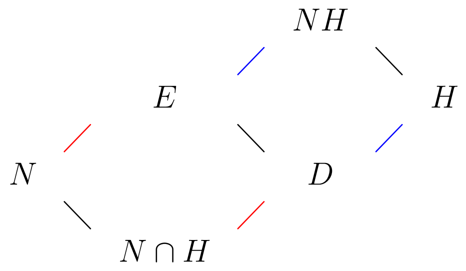
$$H_i \rhd K \rhd H_{i+1}.$$

□

A composition factor is a special case of a *subquotient* of G which is defined to be a quotient of a subgroup of G (i.e., H/N where $N \trianglelefteq H \le G$).

We proved that a finite group has at least one composition series. But there may be several. The Jordan-Hölder Theorem says that the composition factors are uniquely determined up to permutation. The proof uses two of the basic theorems of group theory.

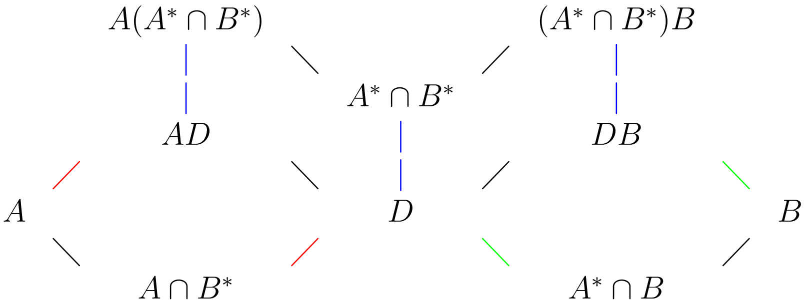
Suppose that $N \trianglelefteq G$ and $H \le G$. Then the Second Isomorphism Theorem says that $NH/N \cong H/N \cap H$. The Correspondence Theorem says that there is a 1-1 correspondence between $\{D \mid N \cap H \trianglelefteq D \trianglelefteq H\}$ and $\{E \mid N \trianglelefteq E \trianglelefteq NH\}$



given by $E = ND$ and $D = E \cap H$ and furthermore,

$$\frac{NH}{E} \cong \frac{H}{D} \qquad \frac{E}{N} \cong \frac{D}{N \cap H}$$

The Zassenhaus Lemma compares two arbitrary subquotients of G, say A^*/A and B^*/B where $G \geq A^* \trianglerighteq A$ and $G \geq B^* \trianglerighteq B$. Taking $H = A^* \cap B^*$ in the Second Isomorphism Theorem and $D = (A \cap B^*)(A^* \cap B)$ in the Correspondence Theorem we get the following diagram.



Since $AD = A(A \cap B^*)(A^* \cap B) = A(A^* \cap B)$ and $DB = (A \cap B^*)(A^* \cap B)B = (A \cap B^*)B$ we get:

Lemma 16.5 (Zassenhaus Lemma).

$$\frac{A(A^* \cap B^*)}{A(A^* \cap B)} \cong \frac{(A^* \cap B^*)B}{(A \cap B^*)B}$$

Theorem 16.6 (Schreier Refinement Theorem). *Any two normal series for G have equivalent refinements where* equivalent *means the sequences of subquotients are isomorphic after permutation.*

Proof. Suppose that

$$G = G_0 \geq G_1 \geq \cdots \geq G_m$$

$$G = H_0 \geq H_1 \geq \cdots \geq H_n$$

are two normal series of G. Then we refine the first series by inserting between each of the n pairs $G_i \geq G_{i+1}$ the following:

$$G_i = G_{i+1}(G_i \cap H_0) \geq G_{i+1}(G_i \cap H_1) \geq \cdots \geq G_{i+1}(G_i \cap H_n) = G_{i+1}$$

Note that the subquotient G_i/G_{i+1} is replaced by the m subquotients

$$\frac{G_{i+1}(G_i \cap H_j)}{G_{i+1}(G_i \cap H_{j+1})} \tag{1}$$

Similarly, the second series can be refined to replace each of the m subquotients H_j/H_{j+1} by the n subquotients

$$\frac{H_{j+1}(G_i \cap H_j)}{H_{j+1}(G_{i+1} \cap H_j)} \tag{2}$$

By the Zassenhaus Lemma (1) is isomorphic to (2) so these are equivalent refinements. □

Theorem 16.7 (Jordan-Hölder Theorem). *Any two composition series for G are equivalent.*

Proof. Delete any repetitions in the refinements given by the Schreier Refinement Theorem. $\square$

16.1 Solvable groups

Definition 16.8. A group G is *solvable* if it has a subnormal series

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = 1$$

where each quotient G_i/G_{i+1} is an abelian group. We will call this a *solvable series.*

For example, any abelian group is solvable even if it is infinite. Another interesting example is the symmetric group S_4 which has the solvable series:

$$S_4 \rhd A_4 \rhd K \rhd 1$$

with quotients $S_4/A_4 \cong \mathbb{Z}/2$, $A_4/K \cong \mathbb{Z}/3$ and $K/1 = K \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

One easy theorem:

Theorem 16.9. *If $H \lhd G$ and both H and G/H are solvable then G is solvable.*

Proof. This is obvious. Let $\phi : G \to Q = G/H$ be the quotient map. Then a solvable series for G is given by:

$$G = \phi^{-1}(Q_0) \geq \phi^{-1}(Q_1) \geq \cdots \geq \phi^{-1}(Q_n) = H = H_0 \geq H_1 \geq \cdots \geq H_m = 1$$

where $Q = Q_0 \geq \cdots \geq Q_n = 1$ and $H = H_0 \geq \cdots \geq H_m = 1$ are solvable series for Q, H. $\square$

Corollary 16.10. *Every finite p-group is solvable.*

Proof. We already proved that every nontrivial finite p-group P has a nontrivial center $Z(P)$ which is abelian and thus solvable. The quotient group $P/Z(P)$ is solvable by induction on $|P|$. So P is solvable by the theorem. $\square$

Corollary 16.11. *If G and H are solvable then so is $G \times H$.*

If G is a solvable group it has a canonical subnormal series with abelian quotients. This is the *derived series* of G:

$$G \geq G' \geq G'' \geq G''' \geq \cdots$$

Recall that the *commutator subgroup* $G' = [G, G]$ is the subgroup of G generated by all commutators $[g, h] = ghg^{-1}h^{-1} = \phi_g(h)h^{-1}$ and $G'' = (G')'$, $G''' = ((G')')'$, etc. We use the shorthand $G^{(n)}$ when this is done n times so, e.g., $G^{(4)} = G''''$. We recall some basic properties of commutator subgroups.

Proposition 16.12. *For any group G we have:*

1. $G' \trianglelefteq G$.[1]
2. G/G' *is abelian. More generally,*
3. G/H *is abelian iff* $H \geq G'$.[2]
4. $H' \leq G'$ *if* $H \leq G$.[3]

We will see later that the derived series is a normal series. However, it is obviously subnormal by 16.12.1 and we have:

Theorem 16.13. *G is solvable if and only if $G^{(n)} = 1$ for some n.*

Proof. It is enough to show that each G_k in a solvable series of G contains $G^{(k)}$. The first step: $G' \leq G_1$ follows from 16.12.3 above. If $G^{(k)} \leq G_k$ then $G^{(k+1)} = (G^{(k)})' \leq (G_k)' \leq G_{k+1}$ by 16.12.3 since G_k/G_{k+1} is abelian. $\square$

Theorem 16.14. *Every subgroup of a solvable group is solvable.*

Proof. By Prop. 16.12.4 we have: $H^{(n)} \leq G^{(n)} \leq 1$. $\square$

Lemma 16.15. *For any homomorphism $f : G \to H$ we have $f(G') = f(G)' \leq H'$. In particular, $\phi(G') = G'$ for any automorphism ϕ of G.*

Proof. f sends the generators of G' to the generators of $f(G)'$ since

$$f([a,b]) = f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} = [f(a), f(b)]$$

$\square$

Theorem 16.16. *Every quotient group of a solvable group is solvable.*

Proof. Let $f : G \to Q$ be an epimorphism. Then $Q^{(n)} = f(G^{(n)}) = 1$. $\square$

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[1] A conjugate of a commutator is a commutator: $x[a,b]x^{-1} = [xax^{-1}, xbx^{-1}]$.

[2] G/H is abelian iff for each a, b we have $Hab = Hba \Leftrightarrow H[a,b] = H \Leftrightarrow [a,b] \in H$.

[3] The generators $[h_1, h_2]$ of H' form a subset of the set of generators of G'.