2 Binomial and complex

This is what I remember saying about binomial coefficients.

Definition 2.1. If n, r are integers with $0 \leq r \leq n$, the *binomial coefficients* $\binom{n}{r}$ are defined to be the coefficients of the polynomial $(1+x)^n$, i.e., they are the integers satisfying the following equation.

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r. \tag{1}$$

In other words, $(1+x)^n$ is the *generating function* for the numbers $\binom{n}{r}$.

One of the things you do with generating functions is to differentiate them:

$$n(1+x)^{n-1} = \sum_{r=0}^{n} r \binom{n}{r} x^{r-1}.$$

Then you can rewrite this as

$$n \sum_{k=0}^{n-1} \binom{n-1}{k} x^k = \sum_{k=0}^{n-1} (k+1) \binom{n}{k+1} x^k.$$

Remember that r is a *dummy variable* so it doesn't mind if you call it k instead.

Comparing the coefficients of x^k we get the following.

Lemma 2.2. *If* $0 \leq k \leq n-1$ *then*

$$n \binom{n-1}{k} = (k+1) \binom{n}{k+1}$$

Theorem 2.3.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof. The proof is by induction on r. Take $x = 0$ in equation (1). Then the equation is $1 = \binom{n}{0}$. Thus the theorem holds for $r = 0$.

Now suppose that $r \geq 0$ and the theorem holds for r. Then

$$\binom{n}{r+1} =_{(a)} \frac{n}{r+1} \binom{n-1}{r} =_{(b)} \frac{n}{r+1} \frac{(n-1)!}{r!(n-1-r)!} =_{(c)} \frac{n!}{(r+1)!(n-r-1)!}$$

where (a) hold by the lemma, (b) holds by induction and (c) follows from the formula $n! = n(n-1)!$. □

Homework: p. 34, # 1.25, 28, 34, 36 (assume # 35).