13 Counting

This section is about the formula for the number of orbits of a group action. The formula is called "Burnside's Lemma" but it is not due to him. (So Rotman calls it "not-Burnsides Lemma.")

Theorem 13.1. *Suppose that a finite group G acts on a finite set X. Then the number of orbits is equal to*

$$N = \frac{1}{|G|} \sum_{g \in G} |F(g)|$$

where $F(g)$ is the set of fixed points of g:

$$F(g) = \{x \in X \mid gx = x\}.$$

During the proof we needed the following lemmas.

Lemma 13.2. *The stabilizer subgroups corresponding to elements of the same orbit are conjugate:*

$$G_y = gG_xg^{-1}$$

if $y = gx$. In particular, $|G_y| = |G_x|$ for all $y \in O(x)$.

Proof. The elements of gG_xg^{-1} are ghg^{-1} where $h \in G_x$ (so $hx = x$). Then

$$(ghg^{-1})y = ghg^{-1}gx = ghx = gx = y$$

so $ghg^{-1} \in G_y$. This proves that $gG_xg^{-1} \subseteq G_y$. The reverse inclusion is similar. □

Lemma 13.3. *The number of orbits is equal to*

$$N = \sum_{x \in X} \frac{1}{|O(x)|}.$$

Proof. Since X is a disjoint union of orbits, the terms in the sum can be collected together:

$$\sum_{x \in X} \frac{1}{|O(x)|} = \sum_{i=1}^{N} \left(\sum_{x \in O(x_i)} \frac{1}{|O(x)|} \right) = \sum_{i=1}^{N} 1 = N$$

where

$$\sum_{x \in O(x_i)} \frac{1}{|O(x)|} = \sum_{x \in O(x_i)} \frac{1}{|O(x_i)|} = 1$$

since there are $|O(x_i)|$ terms and $|O(x)| = |G : G_x| = |G : G_{x_i}| = |O(x_i)|$ by Lemma 13.2. □

Proof of not-Burnside's Lemma. The strategy is to count the same set in two different ways. Let

$$S = \{(g, x) \in G \times X \mid gx = x\}$$

Then

1. $|S| = \sum_{g \in G} (\# \ x \in X \mid gx = x) = \sum_{g \in G} |F(g)|$
2. $|S| = \sum_{x \in X} (\# g \in G \mid gx = x) = \sum_{x \in X} |G_x|$.

The orbit size formula $|O(x)| = |G : G_x| = |G|/|G_x|$ gives us:

$$|G_x| = \frac{|G|}{|O(x)|}$$

So we get:

$$\sum_{g \in G} |F(g)| = |S| = \sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|O(x)|}$$

Divide both sides by $|G|$ to get:

$$\frac{1}{|G|} \sum_{g \in G} |F(g)| = \sum_{x \in X} \frac{1}{|O(x)|} = N$$

by Lemma 13.3. □

Example 13.4. Suppose you have a 4×4 checkerboard square table. You want to color each square either red (R) or black (B). How many ways can this be done if

(a) There is no restriction on the number of red and black squares.

(b) We want an equal number (8R, 8B) of squares of each color.

In this example the group is $G = \mathbb{Z}_4 = \langle g \rangle = \{e, g, g^2, g^3\}$. The set X is the set of all ways to color the table if the table is not allowed to rotate. In the two cases (a), (b) we have:

(a) $|X| = 2^{16}$

(b) $|X| = \binom{16}{8}$.

The number of different patterns up to symmetry is the number of orbits:

$$N = \frac{1}{|G|} \sum_{g \in G} |F(g)| = \frac{1}{4} \left(|F(e)| + |F(g)| + |F(g^2)| + |F(g^3)| \right)$$

The fixed sets are:

1. $F(e) = X$ so $|F(e)| = |X| = 2^{16}$ in case (a) and $|F(e)| = |X| = \binom{16}{8}$ in case (b).

2. The elements of $F(g)$ are given by colorings of the 2×2 squares in one corner so $|F(g)| = 2^4$ in case (a) and $|F(g)| = \binom{4}{2}$ in case (b).

3. The elements of $F(g^2)$ are given by colorings of half of the squares so $|F(g^2)| = 2^8$ in case (a) and $|F(g^2)| = \binom{8}{4}$ in case (b).

4. $g^3 = g^{-1}$ so $|F(g^3)| = |F(g)|$ which is given in (1).

Thus

Case (a) $N = \frac{1}{4}\left(2^{16} + 2^4 + 2^8 + 2^4\right)$

Case (b) $N = \frac{1}{4}\left(\binom{16}{8} + \binom{4}{2} + \binom{8}{4} + \binom{4}{2}\right)$

We decided to apply the formula in a case where we could check the answer.

Example 13.5. Toss two dice. If the dice are different there are $6^2 = 36$ possibilities. This is the size of the set X. If the dice are indistinguishable then we have a symmetry group $G = \mathbb{Z}_2 = \{e, g\}$. The number of different outcomes is given by

$$N = \frac{1}{|G|}\sum_{g \in G} |F(g)| = \frac{1}{2}\left(|F(e)| + |F(g)|\right) = \frac{1}{2}(36 + 6) = \frac{42}{2} = 21$$

which is correct since there are exactly 21 possible outcomes:

11	12	13	14	15	16
22	23	24	25	26	
33	34	35	36		
44	45	46			
55	56				
66					