6 Functions

Functions are written like this:

$$\begin{aligned} f : X &\to Y \\ x &\mapsto f(x) \end{aligned}$$

Functions have the property that for every $x \in X$ there is exactly one $y \in Y$ so that $f(x) = y$. We refer to this property by saying that f is *well defined*. For example the "function" $f(z) = \sqrt{z}$ is not well defined for complex numbers z. (So, it is not a true function.)

Definition 6.1. A function $f : X \to Y$ is called *1-1* or *injective* if it sends distinct elements to distinct elements, i.e., $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

$f : X \to Y$ is called *onto* or *surjective* if every every element of Y is a value of f, i.e., for every $y \in Y$ there is an $x \in X$ so that $f(x) = y$.

$f : X \to Y$ is *bijective* (or *1-1 and onto*) if it satisfies both of these properties.

Theorem 6.2. *A function $f : X \to Y$ is a bijection iff it has an inverse $g = f^{-1} : Y \to X$.*

Proof. I will only show the *necessity* of the condition. ["Necessity" means the first statement implies the second statement.] In other words, I have to show that there exists a two-sided inverse for f ($gf = id_X$ and $fg = id_Y$).

For each $y \in Y$ there is an element $x \in X$ so that $f(x) = y$ (since f is surjective). Let $g(y) = x$.

1. g is well-defined: If x' is another element of X so that $f(x') = y$ then $x = x'$ by injectivity of f.

2. g is a left inverse for f, i.e., $gf = id_X$: Take any $x \in X$. Then $gf(x) = x$ by definition of g.

3. g is a right inverse for f, i.e., $fg = id_Y$: Take any $y \in Y$ then $x = g(y)$ means $f(x) = y$ so $fg(y) = f(x) = y$.

Therefore, g is a well-defined two-sided inverse for f. □

Definition 6.3. For any set X let S_X be the set of all *permutations* of X. These are defined to be bijections $f : X \to X$.

S_X forms a group under composition.