10 Homomorphisms

Definition 10.1. A *homomorphism* $\phi : G \to H$ is a mapping between groups satisfying the condition

$$\phi(gh) = \phi(g)\phi(h)$$

for all $g, h \in G$.

The basic properties of homomorphisms are the following.

1. $\phi(e_G) = e_H$. Proof: Take $g = h = e$ then

   $$\phi(e)e_H = \phi(e) = \phi(ee) = \phi(e)\phi(e)$$

   By cancellation we get $\phi(e) = e$.

2. $\phi(g^{-1}) = \phi(g)^{-1}$. Proof:

   $$e = \phi(e) = \phi(gg^{-1}) = \phi(g)\phi(g^{-1}).$$

3. $\phi(g^n) = \phi(g)^n$. Pf: By induction on n.

More advanced properties of groups depend on the following definition and lemma.

Definition 10.2. A subgroup $N \leq G$ is called *normal* if

$$gNg^{-1} = N$$

for all $g \in G$. The notation is $N \trianglelefteq G$.

Remark 10.3. To show that a subgroup N of G is normal it suffices to show that

$$g \in G, h \in N \Rightarrow ghg^{-1} \in N.$$

This would imply that $gHg^{-1} \subseteq N$ for every $g \in G$ and

$$N = g(g^{-1}Ng)g^{-1} \subseteq gNg^{-1}$$

which implies that $gNg^{-1} = N$.

Lemma 10.4. *If $N \trianglelefteq G$ then $gN = Ng$ for all $g \in G$.*

Theorem 10.5. *If $\phi : G \to H$ is a homomorphism then*

1. $\mathrm{im}(\phi) \leq H$ *(The image of ϕ is a subgroup of H.)*

2. $\ker(\phi) := \{g \in G \,|\, \phi(g) = e\} \trianglelefteq G$. *(The kernel of ϕ is a normal subgroup of G.*

Proof. To show that a subset S of a group G is a subgroup you need to show that it contains e and that

$$g, h \in S \Rightarrow gh^{-1} \in S.$$

The image of ϕ is a subgroup of H since:

1. $\mathrm{im}(\phi)$ contains $e_H = \phi(e_G)$.
2. $\phi(g)\phi(h)^{-1} = \phi(gh^{-1})$.

The kernel of ϕ is a subgroup since it contains e and

$$g, h \in \ker\phi \Rightarrow \phi(g) = \phi(h) = e \Rightarrow \phi(gh^{-1}) = \phi(g)\phi(h)^{-1} = ee^{-1} = e.$$

To prove normality suppose that $h \in \ker\phi$. Then $\phi(h) = e$ so

$$\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g)^{-1} = \phi(g)e\phi(g)^{-1} = e$$

so $ghg^{-1} \in \ker\phi$ for all $g \in G$. So $\ker\phi$ is normal in G by the Remark. □

Examples of homomorphisms:

1. $\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$ where $\mathbb{R}^\times = U(\mathbb{R})$ is the multiplicative group of nonzero real numbers. The kernel of det is $SL_n(\mathbb{R})$.
2. If G is any group and $g \in G$ then the mapping $f : \mathbb{Z} \to G$ given by $\phi(n) = g^n$ is a homomorphism with image $\langle g\rangle$ and kernel $m\mathbb{Z}$ where $m = o(g)$.
3. $sgn : S_n \to \{1, -1\} = U(\mathbb{Z})$

Definition 10.6. An *isomorphism* of groups is a bijective homomorphism $\phi : G \to H$.

Every bijection has an inverse ϕ^{-1}. If f is a homomorphism then so is f^{-1}.

When two groups G, H are isomorphic they have the same "intrinsic" properties. If one is abelian so is the other. The orders of the elements of G are equal to the orders of the corresponding elements of H. In class I pointed out that the groups: $\{even, odd\}$ under addition, $\{1, -1\}$ under multiplication and $\mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\} = \{[0], [1]\}$ are isomorphic.