

7 Permutations

S_n is defined to be the group of all permutations of the set of n “letters” $\{1, 2, \dots, n\}$. Permutations are written in *cycle form*

$$\sigma = (a_1 a_2 \cdots a_k)(b_1 b_2 \cdots b_\ell) \cdots (z_1 z_2 \cdots)$$

where $\sigma(a_1) = a_2, \sigma(a_2) = a_3$, etc. and $\sigma(a_k) = a_1$. Also $\sigma(b_1) = b_2$, etc. We proved in class that the cycles are disjoint as sets.

Lemma 7.1. *If $\alpha, \beta \in S_n$ and $\beta(j) = k$ then the permutation $\alpha\beta\alpha^{-1}$ sends $\alpha(j)$ to $\alpha(k)$.*

Proof. $(\alpha\beta\alpha^{-1})(\alpha(j)) = \alpha(\beta(j)) = \alpha(k)$. □

Theorem 7.2. *The permutation $\alpha\beta\alpha^{-1}$ has the same cycle form as β with each letter replaced by its image under α .*

Proof. If $(x_1 x_2 \cdots x_k)$ is a cycle in the cycle form of β then $\beta(x_i) = x_{i+1}$ and $\beta(x_k) = x_1$. The lemma says that $\alpha\beta\alpha^{-1}$ sends $\alpha(x_i)$ to $\alpha(x_{i+1})$ for $i = 1, \dots, k-1$ and sends $\alpha(x_k)$ to $\alpha(x_1)$ so $(\alpha(x_1) \alpha(x_2) \cdots \alpha(x_k))$ is a cycle in the cycle form of $\alpha\beta\alpha^{-1}$. □

From the example I did in class it is supposed to be clear that the converse of this theorem is also true. Namely, given β and γ with the same cycle form there exists an α so that $\gamma = \alpha\beta\alpha^{-1}$. However, α is not unique.

Definition 7.3. The *order* $o(\sigma)$ of a permutation σ is the smallest positive integer k so that $\sigma^k = 1$. $o(\sigma)$ is defined to be infinity (∞) if no such number k exists.

Proposition 7.4. 1. $\sigma^m = 1$ iff m is a multiple of $o(\sigma)$.

2. $o(\sigma)$ is the least common multiple of the lengths of the cycles in the cycle form of σ .
3. Conjugate elements have the same order.
4. Inverse elements have the same order.

We also proved that S_n is “generated” by the transpositions

$$(12), (23), \dots, (n-1 n).$$