4 Quiz 4

This is a closed book quiz. The time limit is 50 minutes (until the end of class). Answers on page 2.

1) a) Definition: When are two elements of a group G *conjugate* to each other?

b) If $g, h \in G$ are conjugate, prove that they have the same order.

c) Find two permutations with the same order which are not conjugate.

2) a) What is the definition of a coset?

b) If G is a cyclic group of order 6 generated by g how many cosets does $H = \langle g^3 \rangle = \{1, g^3\}$ have and what are they?

3) a) What is the statement of the second isomorphism theorem? (If $H \leq G$ and $N \trianglelefteq G$ then something happens.)

b) Prove that a group of order 70 cannot have more than one normal subgroup of order 7. [Hint: Use the second isomorphism theorem.]

4.1 answers

1) a) Definition: When are two elements of a group G *conjugate* to each other?

Two elements g, h of a group G are conjugate if there exists an $x \in G$ so that $h = xgx^{-1}$.

b) If $g, h \in G$ are conjugate, prove that they have the same order.

Suppose that $g, h \in G$ are conjugate and $o(g) = n$. Then $h = xgx^{-1}$ for some $x \in G$. Thus

$$h^n = (xgx^{-1})^n = xg^nx^{-1} = e$$

This implies that $o(h)$ divides n. Similarly, the order of g divides the order of h. Thus the orders are equal if they are finite. This also shows that if the order of one of the elements is finite, so is the other.

If the orders of g and h are both infinite then $o(g) = \infty = o(h)$. This proves the theorem in all cases.

c) Find two permutations with the same order which are not conjugate.

$(1234)(56)$ and (1234) both have order 4 but they are not conjugate since they have different cycle forms.

2) a) What is the definition of a coset?

Let H be a subgroup of a group G. Then a (left) coset of H in G is defined to be a subset of G of the form

$$gH = \{gh \,|\, h \in H\}$$

for some $g \in G$.

b) If G is a cyclic group of order 6 generated by g how many cosets does $H = \langle g^3 \rangle = \{1, g^3\}$ have and what are they?

There are 3 cosets since

$$|G : H| = \frac{|G|}{|H|} = \frac{6}{2} = 3.$$

They are:

1. $H = \langle g^3 \rangle = \{1, g^3\}$
2. $gH = \{g, g^4\}$
3. $g^2H = \{g^2, g^5\}$.

3) a) What is the statement of the second isomorphism theorem? (If $H \leq G$ and $N \trianglelefteq G$ then something happens.)

If $H \leq G$ and $N \trianglelefteq G$ then $HN \leq G$, $N \trianglelefteq HN$, $N \cap H \trianglelefteq H$ and

$$\frac{HN}{N} \cong \frac{H}{N \cap H}$$

b) Prove that a group of order 70 cannot have more than one normal subgroup of order 7. [Hint: Use the second isomorphism theorem.]

Suppose the statement is not true. Then there are at least two normal subgroups of G of order 7. Call them N, H. By the second isomorphism theorem, NH and $N \cap H$ are both subgroups of G.

First I claim that $N \cap H = \{e\}$. The reason is that, by Lagrange, the order of this subgroup must divide the order of N which is 7. Since 7 is prime, this number must be either 1 or 7. It cannot be 7 since $N \neq H$. Therefore, $|N \cap H| = 1$. Thus

$$\frac{HN}{N} \cong \frac{H}{N \cap H} \cong H$$

has order 7 so

$$|HN| = |N||HN : N| = 7 \cdot 7 = 49$$

This number does not divide 70 so we have a contradiction. (HN is a subgroup of G so its order must divide the order of G.)