11 Quotient Group

Definition 11.1. If N is a normal subgroup of G then the *quotient group* G/N is defined to be the set of cosets

$$G/N = \{gN \mid g \in G\}$$

together with the binary operation given by

$$gN \cdot hN = ghN.$$

The verification of the group laws is straightforward ($N = eN$ is the neutral element, $(gN)^{-1} = g^{-1}N$ and associativity is obvious.) The important point is the meaning of the terms.

The elements of the quotient group G/N are subsets of G. In the notation gN the letter g is ambiguous since it is any element of the coset.

Lemma 11.2 (Gregory's lemma). *$gN = N$ if and only if $g \in N$.*

Proof. Suppose $gN = N$ then $e \in N$ implies that $ge \in gN = N$ so $g \in N$. Conversely, if $g \in N$ then $gN \subseteq N$ since N is closed under multiplication. Also $g^{-1} \in N$ so $g^{-1}N \subseteq N$ which implies that $N \subseteq gN$. So $N = gN$. $\square$

This lemma also tells us when two cosets are equal:

$$gN = hN \quad \Leftrightarrow \quad N = g^{-1}hN \quad \Leftrightarrow \quad g^{-1}h \in N.$$

An example of a quotient group is $\mathbb{Z}/n\mathbb{Z}$. This is the set of congruence classes of integers modulo n. The elements of this set have various names:

$$\overline{k} = [k] = k + n\mathbb{Z}$$

These objects are infinite sets of integers. There are also single elements of the finite set $\mathbb{Z}/n\mathbb{Z}$.

Theorem 11.3 (1st isomorphism theorem). *Suppose that $\phi : G \to H$ is a homomorphism. Then $\ker\phi \trianglelefteq G$, $\operatorname{im}\phi \leq H$ and*

$$G/\ker\phi \cong \operatorname{im}\phi$$

In words: the quotient group $G/\ker\phi$ is isomorphic to the image of ϕ.

Proof. Let $K = \ker\phi$. We already showed that this is a normal subgroup of G. So we just need the isomorphism. An isomorphism is a bijective homomorphism. This will be the mapping $\overline{\phi} : G/\ker\phi \to \operatorname{im}\phi$ given by

$$\overline{\phi}(gK) = \phi(g).$$

There are three things we need to verify

1. $\overline{\phi}$ is *well-defined.*
2. $\overline{\phi}$ is a homomorphism.
3. $\overline{\phi}$ is a bijection.

Well-defined means independent of choices. Here we "chose" g since the same set gK could also have been written $gK = hK$ if $g^{-1}h \in K$. But in that case $\phi(g^{-1}h) = e$ by definition of kernel so

$$\overline{\phi}(hK) = \phi(h) = \phi(gg^{-1}h) = \phi(g)\phi(g^{-1}h) = \phi(g) = \overline{\phi}(gK)$$

This shows that $\overline{\phi}$ is well-defined.

This is obviously a homomorphism:

$$\overline{\phi}(gK \cdot hK) = \overline{\phi}(ghK) = \phi(gh) = \phi(g)\phi(h) = \overline{\phi}(gK)\overline{\phi}(hK).$$

To show that $\overline{\phi}$ is a bijection you have to show it is 1-1 and onto. It is obviously onto since $\operatorname{im}\phi$ is defined to be the set of all $\phi(g)$. To see that it is 1-1 suppose that $\overline{\phi}(gK) = \overline{\phi}(hK)$. Then $\phi(g) = \phi(h)$ so $\phi(g^{-1}h) = e$ so $g^{-1}h \in \ker\phi$ which implies that $gK = hK$. This proves the theorem. $\square$

The example that we did in class was using the determinant homomorphism

$$\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$$

The kernel of det is $SL_n(\mathbb{R})$ by definition. The image is all of $\mathbb{R}^\times$ since any real $x \neq 0$ is the determinant of the diagonal matrix

$$D_x = \begin{pmatrix} x & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

We figured out that the set of all matrices with determinant x was equal to the left coset

$$D_x SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid \det A = x\}$$

Thus $GL_n(\mathbb{R})$ is the disjoint union of the cosets $D_x SL_n(\mathbb{R})$ and the isomorphism

$$\frac{GL_n(\mathbb{R})}{SL_n(\mathbb{R})} \cong \mathbb{R}^\times$$

is given by the 1-1 correspondence $D_x SL_n(\mathbb{R}) \leftrightarrow x$. The matrices D_x give a choice of *representatives* for the cosets of $SL_n(\mathbb{R})$ in $GL_n(\mathbb{R})$.

Theorem 11.4 (2nd isomorphism theorem). *If $N \trianglelefteq G$ and $H \leq G$ then*

1. *NH is a subgroup of G*

2. *$N \cap H \trianglelefteq H$*

3. *$N \trianglelefteq NH$ and*

4. $$\frac{NH}{N} \cong \frac{H}{N \cap H}$$

Proof. Let $\phi : H \to G/N$ be the homomorphism given by $\phi(h) = hN$. Then the kernel is the set of all $h \in H$ so that $\phi(h) = hN = N$. By Gregory's lemma this happens if and only if $h \in N$. But we started with $h \in H$ so:

$$\ker \phi = \{h \in H \mid h \in N\} = N \cap H.$$

The image of ϕ is the set of all cosets $\phi(h) = hN$. By the first isomorphism theorem we have

$$\frac{H}{\ker \phi} = \frac{H}{N \cap H} \cong \operatorname{im} \phi = \{hN \mid h \in H\}$$

On the other hand, HN/N is the group:

$$HN/N = \{hnN \mid h \in H, n \in N\} = \{hN \mid h \in H\}$$

where $hnN = hN$ since $n \in N$. This proves everything except (1) $NH \leq G$ which is "left to the reader." □

Please go over the third isomorphism theorem and the correspondence theorem from the book.