Math 30a Final Exam

This is a take home final. The rules are the same as for homework. You can work together but the words you use to explain your answers must be your own.

Due date: Friday, Dec 15, 2006.

If you are not here you can send the final exam by email (but don't push "reply" you might send it to everyone in the class!). Also, you can send it by fax to the Math Dept at 781-736-3085. You can also fed-ex it directly to me at:

Kiyoshi Igusa
13 Parker Ave
Newton, MA 02459

0.1. **Automorphisms.** One of the main topics of current research is about automorphisms of certain groups, in particular the free group on n generators. Here are some questions about this topic.

0.1.1. *automorphisms of the free group.* Suppose that F_n is the free group on n letters. It is a difficult problem to find all the automorphisms of F_n. But if I give you a specific homomorphism you should be able to show that it is an automorphism.

Here is the specific question. Take F_3 the free group on the letters x, y, z. Let $\phi : F_3 \to F_3$ be the homomorphism which takes x to x, y to a conjugate, say x^5yx^{-5} and takes z to zw where w is some word in the letters x, y and their inverses. You can take $w = x^2y^{-3}xy^4$ if you like to be specific.

a) Show that this is an automorphism of F_3

b) Factor the automorphism as a composition of two automorphisms. [This question concerns the fact that the automorphism ϕ does two things: it conjugates y and it multiplies z by something. It is apparent that this can be done in two steps as $\phi = \phi_2 \circ \phi_1$. Your job is to figure out exactly what those two steps are.]

0.1.2. *invariant and characteristic subgroups.* These are two generalization of the concept of normal subgroup. A subgroup H of G is called *characteristic* if $\phi(H) = H$ for all automorphisms ϕ of G. H is called *invariant* if $\psi(H) \subseteq H$ for all homomorphisms $\psi : G \to G$. These concepts are related by:

$$H \text{ invariant} \Rightarrow H \text{ characteristic} \Rightarrow H \text{ normal}$$

(You don't have to prove that.) Here are some questions about this which you should be able to do.

a) Show that a characteristic subgroup of a normal subgroup is normal. [I.e., if $K \triangleleft H \triangleleft G$ and K is characteristic in H then $K \triangleleft G$.]

b) Show that the center of G is characteristic.

c) Conclude that the center $Z(N)$ of a normal subgroup N of G is normal in G.

d) Find an example of a normal subgroup which is not characteristic. [Hint: Try an abelian group.]

0.2. **Generators and relations.** Properties of groups given by generators and relations are a popular topic of current research. This area is call "combinatorial group theory" and it is studies mainly by topologists. The solution of the Poincaré conjecture by Perelman means that several of the outstanding questions in combinatorial group theory are now settled. But here are some more mundane, simple questions.

0.2.1. *groups generated by involutions.* One of the popular types of groups are those generated by elements of order 2. Let's do $W(A_3)$ (the notation means the *Weyl group* of the *Dynkin diagram* A_3 which has three vertices connected by two edges). This group is:

$$W(A_3) := \langle a, b, c | a^2, b^2, c^2, (ab)^3, (bc)^3, acac \rangle$$

The rule is: one generator x_i for each vertex, $x_i^2 = e$, $(x_i x_j)^3 = e$ if i, j are connected by an edge and $(x_i x_j)^2 = e$ if they are not connected by an edge.

a) Show that there is a homomorphism of $W(A_3)$ onto the symmetric group on 4 letters.

b) Find a similar description for the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. [Hint: it is $W(\Gamma)$ for some graph Γ.]

0.2.2. *Heisenberg group.* This is an important group in physics and in math. (especially if we took real coefficients) There are two descriptions of this group. The notation $H(\mathbb{Z})$ emphasizes that we are only taking integer coordinates.

1) $H(\mathbb{Z})$ is the group of 3×3 upper triangular integer matrices with 1's on the diagonal:

$$H(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \text{ with } x, y, z \in \mathbb{Z} \right\}$$

Call this matrix $M(x, y, z)$.

2) $H(\mathbb{Z}) = \langle a, b, c \,|\, [a, c], [b, c], abc^{-1}a^{-1}b^{-1} \rangle$ where $[x, y] := xyx^{-1}y^{-1}$.

Your job is to show that these two descriptions are equivalent. This may be difficult for you. So, here are step by step instructions.

(1) Call the second group H_2 and the first group H_1.
(2) Find a homomorphism $\phi : H_2 \to H_1$ so that $\phi(a) = M(1, 0, 0)$, $\phi(b) = M(0, 1, 0)$, $\phi(c) = M(0, 0, 1)$. You have to show that you get a homomorphism. Use the universal property.
(3) Let $Z_1 = \{M(0, 0, z) \,|\, z \in \mathbb{Z}\}$. Show that this is a normal subgroup of H_1.
(4) Let $Z_2 = \langle c \rangle$. Show that this is a normal subgroup of H_2.
(5) Show that ϕ maps Z_2 isomorphically onto Z_1.
(6) What are the quotient groups H_1/Z_1 and H_2/Z_2?
(7) What is the induced homomorphism $\overline{\phi} : H_2/Z_2 \to H_1/Z_1$?
(8) Show that $\overline{\phi}$ is an isomorphism.
(9) Conclude (by the theorem below) that $\phi : H_2 \to H_1$ is an isomorphism.

You need the following theorem which I will hopefully in class on the last day. But you can use it anyway if I don't get to it:

Theorem 0.1. *Suppose G_1, G_2 are groups with normal subgrouops $N_1 \lhd G_1$ and $N_2 \lhd G_2$. Suppose that $\phi : G_1 \to G_2$ is a homomorphism so that $\phi(N_1) \subseteq N_2$. Then ϕ is an isomorphism if and only if*

(1) *$\phi : N_1 \to N_2$ is an isomorphism and*
(2) *the induced homomorphism $\overline{\phi} : G_1/N_1 \to G_2/N_2$ is an isomorphism.*

0.3. **Quantum groups.** This is a very popular topic. Quantum algebra is a recent area of math which has generated lots of literature. I can't think of any simple questions about quantum groups that I can ask you since it would take forever just to define the terms. However, I can ask you about *quantum integers*.

If $\hbar$ is Plank's constant (a very small number) then $q = e^{i\hbar}$ is a complex number which is very close to 1. Usually we think of it as a formal variable. But on this test you are to take it as a complex number. (It is approximately $1 + i\hbar$.) One definition of the quantum integer n is

$$[n] := \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}$$

This is very close to being equal to n.

These quantum integers don't quite add up the way they should:

$$[n] + [m] \neq [n + m] \tag{0.1}$$

Your job is to fix this problem.

(1) The first step is to quantify the problem. The equation (0.1) says that something is not a homomorphism. Make a precise statement: $\phi : G \to H$ defined by ... is not a homomorphism.
(2) To fix the problem we need to construct a *semidirect product*. This will be: $G = \mathbb{C} \rtimes \mathbb{Z}$ where the action of the integers on complex numbers is given by $n \cdot z = q^n z$. Verify that this is an action.
(3) What is the identity in $\mathbb{C} \rtimes \mathbb{Z}$?
(4) What is the inverse of (z, n)?
(5) Show that $\phi : \mathbb{Z} \to \mathbb{C} \rtimes \mathbb{Z}$ given by

$$\phi(n) = ([n], n)$$

is a homomorphism.