

MATH 30A FINAL EXAM ANSWERS

Here are my answers to the take home final. The rules are the same as for homework. You can work together but the words you use to explain your answers must be your own.

Due date: Friday, Dec 15, 2006.

0.1. **Automorphisms.**

0.1.1. *automorphisms of the free group.* Take F_3 the free group on the letters x, y, z . Let $\phi : F_3 \rightarrow F_3$ be the homomorphism which takes x to x , y to a conjugate, say x^5yx^{-5} and takes z to zw where w is some word in the letters x, y and their inverses. You can take $w = x^2y^{-3}xy^4$ if you like to be specific.

a) Show that this is an automorphism of F_3

The easiest method is to do part (b) first (since you have to do that anyway). But, if this were the only question, the best way is to write down the inverse $\phi^{-1} = \psi$. By the universal property ψ is determined by $\psi(x), \psi(y), \psi(z)$:

- (1) Let $\psi(x) = x$. Then $\phi(\psi(x)) = \phi(x) = x$ and $\psi(\phi(x)) = \psi(x) = x$.
- (2) Let $\psi(y) = x^{-5}yx^5$. Then $\phi(\psi(y)) = \phi(x^{-5}yx^5) = y$ and $\psi(\phi(y)) = y$.
- (3) Let $\psi(z) = z\psi(w)^{-1}$. Here I used the fact that w is a word in x, y (and inverses) and ψ is already defined for x, y and therefore, by the homomorphism property (OP), it is defined on w . Now we have to check that $\phi(\psi(z)) = z = \psi(\phi(z))$:

$$\phi(\psi(z)) = \phi(z)\phi(\psi(w))^{-1} = (zw)w^{-1} = z$$

$$\psi(\phi(z)) = \psi(zw) = \psi(z)\psi(w) = z\psi(w)^{-1}\psi(w) = z$$

We have defined ψ and shown that $\phi \circ \psi = id = \psi \circ \phi$. Therefore, ψ is the inverse of ϕ and ϕ is an isomorphism. (To be really precise you would say: $\phi \circ \psi = id$ implies ϕ is onto and $\psi \circ \phi = id$ implies ϕ is 1-1.)

b) Factor the automorphism as a composition of two automorphisms.

The words give it away: ϕ_1 is the automorphism of F_3 which conjugates y and ϕ_2 is the automorphism of F_3 which multiplies z by some word.

- (1) $\phi_1(x) = x, \phi_1(y) = x^5yx^{-5}, \phi_1(z) = z$. This is an automorphism of F_3 with inverse $\psi_1(x) = x, \psi_1(y) = x^{-5}yx^5, \psi_1(z) = z$.
- (2) $\phi_2(x) = x, \phi_2(y) = y, \phi_2(z) = zw$. This is an automorphism of F_3 with inverse $\psi_2(x) = x, \psi_2(y) = y, \psi_2(z) = zw^{-1}$.
- (3) The composition $\phi_2 \circ \phi_1$ sends x to x , y to x^5yx^{-5} and z to zw . So, $\phi_2 \circ \phi_1 = \phi$.
- (4) Some students took another automorphism ϕ_0 given by $\phi_0(x) = x, \phi_0(y) = y, \phi_0(z) = z\psi_1(w)$. This is an automorphism of F_3 with inverse $\psi_0(x) = x, \psi_0(y) = y, \psi_0(z) = z\psi_1(w)^{-1}$. Then $\phi_1 \circ \phi_0 = \phi$. For example

$$\phi_1 \circ \phi_0(z) = \phi_1(z\psi_1(w)) = \phi_1(z)\phi_1(\psi_1(w)) = zw = \phi(z)$$

Once you factor ϕ as a composition of two automorphisms $\phi = \phi_2 \circ \phi_1$, you get its inverse as a composition of inverses:

$$\phi^{-1} = (\phi_2 \circ \phi_1)^{-1} = \psi_1 \circ \psi_2$$

This shows that ϕ is invertible and thus an automorphism. (So, I didn't have to do part (a).)

0.1.2. *invariant and characteristic subgroups.* These are two generalization of the concept of normal subgroup. A subgroup H of G is called *characteristic* if $\phi(H) = H$ for all automorphisms ϕ of G . H is called *invariant* if $\psi(H) \subseteq H$ for all homomorphisms $\psi : G \rightarrow G$. These concepts are related by:

$$H \text{ invariant} \Rightarrow H \text{ characteristic} \Rightarrow H \text{ normal}$$

(You don't have to prove that.) Here are some questions about this which you should be able to do.

a) Show that a characteristic subgroup of a normal subgroup is normal. [I.e., if $K \triangleleft H \triangleleft G$ and K is characteristic in H then $K \triangleleft G$.]

For any $g \in G$ we have $gHg^{-1} = H$. This means that conjugation by g ($\phi_g(x) = gxg^{-1}$) gives an automorphism ϕ_g of H . Since K is characteristic in H , $\phi_g(K) = K$. Since this holds for all $g \in G$, K is normal in G .

b) Show that the center of G is characteristic.

Suppose that ϕ is an automorphism of G . Then ϕ has an inverse ψ . We want to show that $\phi(Z(G)) = Z(G)$.

Suppose that $z \in Z(G)$ and $g \in G$. Since z is central we have

$$z\psi(g) = \psi(g)z$$

Apply ϕ to both sides to get:

$$\phi(z)g = g\phi(z)$$

Therefore, $\phi(z)$ is central. This shows $\phi(Z(G)) \subseteq Z(G)$. A similar argument shows $\psi(Z(G)) \subseteq Z(G)$. So, $Z(G) \supseteq \phi(Z(G))$ and we get $\phi(Z(G)) = Z(G)$.

c) Conclude that the center $Z(N)$ of a normal subgroup N of G is normal in G .

This is obvious.

d) Find an example of a normal subgroup which is not characteristic. [Hint: Try an abelian group.]

The smallest example is $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then $H = \mathbb{Z}_2 \oplus 0$ is a normal subgroup which is not characteristic since there is an automorphism of G which switches the two factors: $\phi(x, y) = (y, x)$. Then $\phi(H) = 0 \oplus \mathbb{Z}_2 \neq H$.

Some students took \mathbb{Z}_n for various n . This was not a good choice because: *Every subgroup of a cyclic group is characteristic!* (If $H \leq \mathbb{Z}_n$ then $\phi(H) = H$ for any automorphism ϕ because there is only one subgroup of \mathbb{Z}_n with the same order as H .)

0.2. Generators and relations.

0.2.1. *groups generated by involutions.* One of the popular types of groups are those generated by elements of order 2. Let's do $W(A_3)$ (the notation means the *Weyl group* of the *Dynkin diagram* A_3 which has three vertices connected by two edges). This group is:

$$W(A_3) := \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (bc)^3, acac \rangle$$

The rule is: one generator x_i for each vertex, $x_i^2 = e$, $(x_i x_j)^3 = e$ if i, j are connected by an edge and $(x_i x_j)^2 = e$ if they are not connected by an edge.

a) Show that there is a homomorphism of $W(A_3)$ onto the symmetric group on 4 letters.

Let $\phi : W(A_3) \rightarrow S_4$ be given by

$$\phi(a) = (12), \quad \phi(b) = (23), \quad \phi(c) = (34)$$

Then obviously, $\phi(a)^2 = \phi(b)^2 = \phi(c)^2 = e$ and $\phi(a)\phi(c) = \phi(c)\phi(b)$. So, we just have to check that ϕ takes ab and bc to elements of order 3.

$$\phi(a)\phi(b) = (12)(23) = (123), \quad \phi(b)\phi(c) = (23)(34) = (234)$$

These obviously have order 3. So, ϕ takes each of the relations of $W(A_3)$ to e . Therefore, ϕ is a homomorphism (by the universal property). The transpositions (12), (23), (34) generate S_4 . Therefore $\phi : W(A_3) \rightarrow S_4$ is onto.

b) Find a similar description for the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. [Hint: it is $W(\Gamma)$ for some graph Γ .]

Let Γ be the graph with two vertices and no edges. Each vertex gives a generator of order 2 and the lack of edges means they commute. So,

$$W(\Gamma) = \langle a, b | a^2, b^2, (ab)^2 \rangle$$

In $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $x = (1, 0)$ and $y = (0, 1)$ have order 2 and commute. So, there is a homomorphism $\phi : W(\Gamma) \rightarrow G$ given by $\phi(a) = x, \phi(b) = y$. The inverse $\psi : G \rightarrow W(\Gamma)$ is given by $\psi(n, m) = a^n b^m$.

0.2.2. Heisenberg group. 1) $H(\mathbb{Z})$ is the group of 3×3 upper triangular integer matrices with 1's on the diagonal:

$$H_1 = H(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \text{ with } x, y, z \in \mathbb{Z} \right\}$$

Call this matrix $M(x, y, z)$.

2) $H_2 = H(\mathbb{Z}) = \langle a, b, c | [a, c], [b, c], abc^{-1}a^{-1}b^{-1} \rangle$ where $[x, y] := xyx^{-1}y^{-1}$.

Your job is to show that these two descriptions are equivalent. This may be difficult for you. So, here are step by step instructions.

1) Call the second group H_2 and the first group H_1 .

OK, I did it.

2) Find a homomorphism $\phi : H_2 \rightarrow H_1$ so that $\phi(a) = M(1, 0, 0), \phi(b) = M(0, 1, 0), \phi(c) = M(0, 0, 1)$. You have to show that you get a homomorphism. Use the universal property.

Specifying $\phi(a), \phi(b), \phi(c)$ as I did gives a homomorphism if and only if the three relations are satisfied.

- $[\phi(a), \phi(c)] = e$: $\phi(a)$ and $\phi(c)$ need to commute. But $\phi(c) = M(0, 0, 1)$ is central in H_1 . So, it commutes with everything.
- $[\phi(b), \phi(c)] = e$ since $\phi(c)$ is central.
- $\phi(a)\phi(b)\phi(c)^{-1}\phi(a)^{-1}\phi(b)^{-1} = e$ This is the same as $\phi(a)\phi(b) = \phi(b)\phi(a)\phi(c)$

$$\phi(a)\phi(b) = M(1, 0, 0)M(0, 1, 0) = M(1, 1, 1)$$

$$\phi(b)\phi(a)\phi(c) = M(0, 1, 0)M(1, 0, 0)M(0, 0, 1) = M(1, 1, 0)M(0, 0, 1) = M(1, 1, 1)$$

3) Let $Z_1 = \{M(0, 0, z) | z \in \mathbb{Z}\}$. Show that this is a normal subgroup of H_1 .

Z_1 is the cyclic subgroup generated by $M(0, 0, 1)$. Since this is central, Z_1 is central (contained in the center) and therefore normal.

4) Let $Z_2 = \langle c \rangle$. Show that this is a normal subgroup of H_2 .

Since c commutes with the other two generators, Z_2 is central and thus normal.

5) Show that ϕ maps Z_2 isomorphically onto Z_1 .

⁴ $\phi(c) = M(0, 0, 1)$. So, $\phi(c^n) = M(0, 0, 1)^n = M(0, 0, n)$. This gives a bijection from Z_2 to Z_1 . We already proved (in step 1) that ϕ is a homomorphism. So, $\phi : Z_2 \rightarrow Z_1$ is an isomorphism.

6) What are the quotient groups H_1/Z_1 and H_2/Z_2 ?

Since $abZ_2 = baZ_2$, the elements of H_2/Z_2 can be written uniquely as $a^n b^m Z_2$:

$$H_2/Z_2 = \{a^n b^m Z_2 \mid n, m \in \mathbb{Z}\}$$

If x_1, y_1, z_1 are fixed then

$$M(x_1, y_1, z_1)Z_1 = \{M(x_1, y_1, z_1 + z) \mid z \in \mathbb{Z}\}$$

Thus, in each coset, the x and y coordinates are fixed and there is a unique representative with z coordinate equal to zero. So,

$$H_1/Z_1 = \{M(x, y, 0)Z_1 \mid x, y \in \mathbb{Z}\}$$

7) What is the induced homomorphism $\bar{\phi} : H_2/Z_2 \rightarrow H_1/Z_1$?

We have a homomorphism $\phi_1 : H_2 \rightarrow H_1/Z_1$ given by

$$\phi_1(w) = \phi(w)Z_1$$

Since $f(Z_2) \subseteq Z_1$ we get an induced homomorphism $\bar{\phi} : H_2/Z_2 \rightarrow H_1/Z_1$ given by

$$\bar{\phi}(wZ_2) = \phi(w)Z_1.$$

8) Show that $\bar{\phi}$ is an isomorphism.

First, $\bar{\phi}$ is onto since any $M(x, y, 0)Z_1 \in H_1/Z_1$ is the image of $a^x b^y Z_2$. Also, $\bar{\phi}(a^n b^m Z_2) = M(n, m, 0)Z_1$ is the identity Z_1 only if $M(x, y, 0) \in Z_1$ iff $n = m = 0$. So, the kernel of $\bar{\phi}$ is trivial and $\bar{\phi}$ is 1-1.

9) Conclude (by the theorem below) that $\phi : H_2 \rightarrow H_1$ is an isomorphism.

Yes, that is what the theorem says.

Theorem 0.1. Suppose G_1, G_2 are groups with normal subgroups $N_1 \triangleleft G_1$ and $N_2 \triangleleft G_2$. Suppose that $\phi : G_1 \rightarrow G_2$ is a homomorphism so that $\phi(N_1) \subseteq N_2$. Then ϕ is an isomorphism if and only if

(1) $\phi : N_1 \rightarrow N_2$ is an isomorphism and

(2) the induced homomorphism $\bar{\phi} : G_1/N_1 \rightarrow G_2/N_2$ is an isomorphism.

0.3. Quantum groups. If \hbar is Planck's constant (a very small number) then $q = e^{i\hbar}$ is a complex number which is very close to 1. Usually we think of it as a formal variable. But on this test you are to take it as a complex number. (It is approximately $1 + i\hbar$.) One definition of the quantum integer n is

$$[n] := \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}$$

This is very close to being equal to n .

These quantum integers don't quite add up the way they should:

$$(0.1) \quad [n] + [m] \neq [n + m]$$

Your job is to fix this problem.

1) The first step is to quantify the problem. The equation (0.1) says that something is not a homomorphism. Make a precise statement: $\phi : G \rightarrow H$ defined by ... is not a homomorphism.

$\phi : G \rightarrow H$ defined by $\phi(n) = [n]$ where G, H are the additive groups $G = \mathbb{Z}$ and $H = \mathbb{C}$ is not a homomorphism.

Or, you could just write:

$\phi : (\mathbb{Z}, +) \rightarrow (\mathbb{C}, +)$ defined by $\phi(n) = [n]$ is not a homomorphism.

2) To fix the problem we need to construct a *semidirect product*. This will be: $G = \mathbb{C} \rtimes \mathbb{Z}$ where the action of the integers on complex numbers is given by $n \cdot z = q^n z$. Verify that this is an action.

This is an action because $0 \cdot z = q^0 z = z$ and

$$n \cdot (m \cdot (z)) = n \cdot (q^m z) = q^n q^m z = q^{n+m} z = (n + m) \cdot z$$

3) What is the identity in $\mathbb{C} \rtimes \mathbb{Z}$?

The operation is

$$(z, n)(x, m) = (z + q^n x, n + m)$$

This is equal to (x, m) if $z = n = 0$. So, $(0, 0)$ is the identity.

4) What is the inverse of (z, n) ?

We have to solve the equation

$$(z, n)(x, m) = (z + q^n x, n + m) = (0, 0)$$

The solution is $m = -n$ and $q^n x = -z$ or $x = -q^{-n} z$. So,

$$(z, n)^{-1} = (-q^{-n} z, -n)$$

5) Show that $\phi : \mathbb{Z} \rightarrow \mathbb{C} \rtimes \mathbb{Z}$ given by

$$\phi(n) = ([n], n)$$

is a homomorphism.

$$\phi(n)\phi(m) = ([n], n)([m], m) = ([n] + q^n [m], n + m) = ([n + m], n + m) = \phi(n + m)$$

since

$$[n] + q^n [m] = \frac{1 - q^n}{1 - q} + q^n \left(\frac{1 - q^m}{1 - q} \right) = \frac{1 - q^n}{1 - q} + \frac{q^n - q^n q^m}{1 - q} = \frac{1 - q^{n+m}}{1 - q} = [n + m]$$

Therefore, ϕ is a homomorphism.