

## MATH 30A, HOMEWORK 2

p. 67 # 2,3,4,7,11,15,16,22

2.  $Q = (\mathbb{Q}, +)$ ,  $Q' = (\mathbb{Q}_{\neq 0}, \cdot)$ . In  $Q$  list the elements of  $\langle \frac{1}{2} \rangle$ . Do the same for  $Q'$ .

In  $Q = (\mathbb{Q}, +)$ ,  $\langle \frac{1}{2} \rangle$  is the *additive* subgroup generated by  $\frac{1}{2}$ . Thus it is the set of all multiples of  $1/2$ . These are called *half integers*.

$$\left\langle \frac{1}{2} \right\rangle = \left\{ \frac{n}{2} \mid n \in \mathbb{Z} \right\} = \left\{ \dots, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \right\}$$

In  $Q' = (\mathbb{Q}_{\neq 0}, \cdot)$ ,  $\langle \frac{1}{2} \rangle$  is the *multiplicative* group generated by  $\frac{1}{2}$ . This is the set of all powers of  $\frac{1}{2}$  which is the same as the set of all powers of 2:

$$\left\langle \frac{1}{2} \right\rangle = \{2^n \mid n \in \mathbb{Z}\} = \left\{ \dots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \dots \right\}$$

4. Prove that  $g$  and  $g^{-1}$  have the same order.

We know that  $(ab)^{-1} = b^{-1}a^{-1}$ . By induction on  $n$  it follows that

$$(g_1 g_2 \cdots g_n)^{-1} = g_n^{-1} \cdots g_2^{-1} g_1^{-1}$$

When all the  $g_i$ 's are the same this gives Lemma:  $(g^n)^{-1} = (g^{-1})^n$ .

If  $|g| = n$  then  $g^n = e$ . Therefore, by the lemma,  $(g^{-1})^n = (g^n)^{-1} = e^{-1} = e$ . So,  $|g^{-1}| \leq |g|$ . This formula, when applied to  $g^{-1}$  gives  $|g| \leq |g^{-1}|$ . So,  $|g| = |g^{-1}|$ .

16. Prove that  $C(a) = C(a^{-1})$ .

Suppose that  $b \in C(a)$ . Then  $ab = ba$ . Multiply on the left and right by  $a^{-1}$ :

$$a^{-1}aba^{-1} = a^{-1}baa^{-1}$$

Since  $a^{-1}a = e = aa^{-1}$ , this gives  $ba^{-1} = a^{-1}b$ . In other words,  $b \in C(a^{-1})$ . Therefore,  $C(a) \subseteq C(a^{-1})$ . Replace  $a$  with  $a^{-1}$  and we get

$$C(a^{-1}) \subseteq C((a^{-1})^{-1}) = C(a)$$

Therefore,  $C(a) = C(a^{-1})$ . [Two sets are equal if each is contained in the other.]

22. Must the center of a group be abelian?

Yes. Elements of the center of  $G$  commute with all elements of  $G$ , a fortiori they commute with each other.

p. 82 #9,10,22,30,54,63

**10.** Let  $G = \langle a \rangle$  and  $|a| = 24$ . List the generators of the subgroup of order 8.

The subgroup of order 8 is generated by  $a^{24/8} = a^3$  and by  $a^9, a^{15}, a^{21}$  since 1, 3, 5, 7 are the units modulo 8.

**22.** Prove that a group of order 3 must be cyclic.

Suppose that  $G$  is a group of order 3. One of the elements is  $e$ . So,  $G = \{e, a, b\}$ . We want to show that  $a^2 = b$  and  $a^3 = e$ . But these are not that obvious to prove. The thing that we can prove is that

$$ab = e$$

Proof: It can't be anything else! If  $ab = a$  then  $b = e$ , a contradiction. If  $ab = b$  then  $a = e$ , a contradiction. Next,

$$a^2 = b$$

because it can't be anything else! If  $a^2 = e$  then  $a^2 = ab$  would give  $a = b$ , a contradiction. If  $a^2 = a$  then  $a = e$ , a contradiction. Finally,

$$a^3 = aa^2 = ab = e$$

and we are done.

**30.** Suppose that  $a$  has odd order and  $aba^{-1} = b^{-1}$ . Show that  $b^2 = e$ .

This is hard. First, you need to show that  $a^2ba^{-2} = b$ :

$$a^2ba^{-2} = a(aba^{-1})a^{-1} = ab^{-1}a^{-1} = (aba^{-1})^{-1} = (b^{-1})^{-1} = b$$

Multiplying on the right by  $a^2$  gives:  $a^2b = ba^2$ . In other words,  $a^2$  commutes with  $b$ . By induction this implies that any even power of  $a$  commutes with  $b$ :

$$a^{2k}b = a^2a^{2k-2}b = a^2ba^{2k-2} = ba^2a^{2k-2} = ba^{2k}$$

We know that  $|a| = 2k + 1$ . So,

$$b = a^{2k+1}ba^{-2k-1} = aa^{2k}ba^{-2k}a^{-1} = aba^{-1} = b^{-1}$$

So,  $b^2 = e$ .

**54.** If  $|a|, |b|$  are relatively prime then show that  $\langle a \rangle \cap \langle b \rangle = \{e\}$ .

First note that  $H = \langle a \rangle \cap \langle b \rangle$  is a subgroup of  $\langle a \rangle$  and of  $\langle b \rangle$ . So, by Theorem 4.3,  $|H|$  divides  $|\langle a \rangle| = |a|$  and it divides  $|\langle b \rangle| = |b|$ . If these numbers are relatively prime then  $|H|$  must be 1. So,  $H = \langle a \rangle \cap \langle b \rangle = \{e\}$ .

p. 112 #4,7,8,23,25,26,45

4. What are the orders of the following permutations?

a.  $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{bmatrix}$

In cycle notation this is  $(12)(356)$  which has order  $\text{lcm}(2, 3) = 6$ .

b.  $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

In cycle form this is  $(1753)(264)$ . [It would be a 7-cycle if 6 and 7 were switched in the array notation.] So the order is  $\text{lcm}(4, 3) = 12$ .

8. What is the maximum order of any element of  $A_{10}$ ?

There are 42 possible cycles forms for the permutations of 10. (OK, I cheated. I used a computer.). They are:

**odd:** 10, 8, 6, 4, 2, 7-2, 6-3, 5-4, 5-2, 4-3, 3-2, 6-2-2, 5-3-2, 4-4-2, 4-3-3, 4-2-2, 3-3-2, 2-2-2, 3-2-2-2, 2-2-2-2-2.

**even:** 9, 7, 5, 3, 1, 8-2, 7-3, 6-4, 5-5, 6-2, 5-3, 4-4, 4-2, 3-3, 2-2, 5-2-2, 4-3-2, 3-3-3, 3-2-2, 4-2-2-2, 3-3-2-2, 2-2-2-2-2.

The orders for the even ones are

$$9, 7, 5, 3, 1, 8, 21, 12, 5, 6, 15, 4, 4, 3, 2, 10, 12, 3, 6, 4, 6, 2$$

respectively. The maximum is 21.

26. Prove that  $(1234)$  is not the product of 3-cycles.

This is easy.  $(1234) = (12)(23)(34)$  is an odd permutation so it cannot be a product even permutations. (3-cycles are even.)