

## MATH 30A, HOMEWORK 2

p. 67 # 2,3,4,7,11,15,16,22

3. Find the order of each element in  $Q$  and  $Q'$  in problem 2.

The elements of  $Q$  and  $Q'$  have infinite order with three exceptions: 0 has order 1 in  $Q$ , 1 has order 1 in  $Q'$  and  $-1$  has order 2 in  $Q'$ .

7. Show that if  $a \in G$  then  $|a| \leq G$ .

Suppose that  $|a| = n$ . Then the subgroup  $\langle a \rangle$  of  $G$  has  $n$  elements. So  $G$  has at least  $n$  elements.

11. Find a group  $G$  containing elements  $a, b$  with order 2 so that  $ab$  has order  $n \geq 3$ .

The dihedral group  $D_n$  has a rotation  $R$  of order  $n$ . Let  $a$  be any reflection. Then  $b = aR$  is also a reflection (a reflection times a rotation is a reflection in any dihedral group). So,  $a, b$  have order 2 and  $ab = a^2R = R$  has order  $n$ .

15. Show that the center of  $G$  is the intersection of all centralizers:

$$Z(G) = \bigcap_{a \in G} C(a)$$

This is obvious.  $g \in C(a)$  iff  $g$  commutes with  $a$ . So,  $g \in \bigcap C(a)$  iff it commutes with every  $a \in G$ . But this is the same as  $g \in Z(G)$  by definition of the center of  $G$ .

p. 82 #9,10,22,30,54,63

9, corrected. How many subgroups does  $\mathbb{Z}_{20}$  have? List the generators of each of these subgroups. Suppose that  $G = \langle a \rangle$  and  $|a| = 20$ . Then how many subgroups does  $G$  have? List the generators of each of these subgroups.

$\mathbb{Z}_{20}$  has four subgroups:

$$H_1 = \{0\} = \langle 0 \rangle$$

$$H_2 = \{0, 10\} = \langle 10 \rangle$$

$$H_4 = \{0, 5, 10, 15\} = \langle 5 \rangle = \langle 15 \rangle$$

$$H_5 = \{0, 4, 8, 12, 16\} = \langle 4 \rangle = \langle 8 \rangle = \langle 12 \rangle = \langle 16 \rangle$$

$$H_{10} = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\} = \langle 2 \rangle = \langle 6 \rangle = \langle 14 \rangle = \langle 18 \rangle$$

and the whole group

$$\mathbb{Z}_{20} = \langle 1 \rangle = \langle 3 \rangle = \langle 7 \rangle = \langle 9 \rangle = \langle 11 \rangle = \langle 13 \rangle = \langle 17 \rangle = \langle 19 \rangle$$

$G = \langle a \rangle$  has six subgroups  $H_1, H_2, H_4, H_5, H_{10}, H_{20}$  (where the subscript denotes the number of elements) with generators  $e$  for  $H_1$ ,  $a^{10}$  for  $H_2$ ,  $a^5$  and  $a^{15}$  for  $H_4$ ,  $a^4, a^8, a^{12}, a^{16}$  for  $H_{10}$  and  $a, a^3, a^7, a^9, a^{11}, a^{13}, a^{17}, a^{19}$  for  $G = H_{20}$ .

**63.** Let  $a, b \in G$  If  $|a| = 12, |b| = 22$  and  $\langle a \rangle \cap \langle b \rangle \neq \{e\}$ , prove that  $a^6 = b^{11}$ .

$H = \langle a \rangle \cap \langle b \rangle$  is a subgroup of the cyclic group  $\langle a \rangle$ . So,  $H$  is cyclic and  $H = \langle g \rangle$  for some  $g \in H$ . We are assuming that  $H$  is nontrivial (having more than one element). So,  $g \neq e$ . Since  $g \in \langle a \rangle$ ,  $g = a^n$  for some  $n$  and the order of  $g$  divides 12. Since  $g \in \langle b \rangle$ ,  $g = b^m$  for some  $m$  and the order of  $g$  divides 22. But the only numbers dividing 12 and 22 are 1 and 2. So, the order of  $g$  must be 2. But the only power of  $a$  with order 2 is  $a^{12/2} = a^6$  and the only power of  $b$  with order 2 is  $b^{22/2} = b^{11}$ . So,  $g = a^6 = b^{11}$ .

p. 112 #4,7,8,23,25,26,45

**7.** What are the possible orders for the elements of  $S_6$  and  $A_6$ ? What about  $S_7, A_7$ ?

The elements of  $S_6$  can have order 1,2,3,4,5,6 (take a cycle of that length). But the elements of  $A_6$  can only have orders 1,2,3,4,5. Three and five cycles are even with orders 3,5. The permutations (12)(34) and (12)(3456) are even with orders 2,4.

The group  $S_7$  has elements of order 1 through 7 given by cycles and it has elements of order 10 and 12 given by (12)(34567) and (123)(4567). These are odd permutations which do not lie in  $A_7$ . The group  $A_7$  has elements of order 1 through 7 only. Elements of order 3,5,7 are given by cycles. Elements of order 2,4,6 are given by (12)(34), (12)(3456), (12)(34)(567) respectively.

**23, corrected.** Use Table 5.1 to compute the following. The centralizer of  $\alpha_3 = (13)(24)$  in  $A_4$ , the centralizer of  $\alpha_{12} = (124)$ .

$C(\alpha_3) = \{e, (12)(34), (14)(23), (13)(24)\}$ ,  $C(\alpha_{12}) = \{e, (124), (142)\}$

**25.** How many odd permutations of order 4 does  $S_6$  have?

An odd permutation of order 4 in  $S_6$  must be a 4-cycle and there are  $6 \cdot 5 \cdot 4 \cdot 3/4 = 90$  four-cycles in  $S_6$ .

**45.** Show that every element of  $A_n$  is a 3-cycle or product of 3-cycles.

By definition every element of  $A_n$  is a product of an even number of transpositions. So, it suffices to show that any product of two transpositions  $(ab)(cd)$  is a 3-cycle or product of 3-cycles.

First, if the two transpositions overlap, the product is a 3-cycle since  $(ab)(bc) = (abc)$ .

If they don't overlap then  $(ab)(cd) = (ab)(bc)(bc)(cd) = (abc)(bcd)$  is a product of 3-cycles.