MATH 30A, HOMEWORK 3

p. 148 # 6,12, 21, 22, 25, 30.

6. Let n be a positive integer. Let $H = n\mathbb{Z}$. Find all left cosets of H in $\mathbb{Z}$. How many are there?

There are n cosets: $n\mathbb{Z}, n\mathbb{Z}+1, n\mathbb{Z}+2, \cdots, n\mathbb{Z}+n-1$. Here

$$n\mathbb{Z}+k = \{nx+k \,|\, x \in \mathbb{Z}\} = \{y \in \mathbb{Z} \,|\, y \equiv k \mod n\}$$

12. Let $\mathbb{C}^*$ be the group of nonzero complex numbers under multiplication and let $H = \{a+bi \in \mathbb{C}^* \,|\, a^2+b^2 = 1\}$. Give a geometric description of the cosets of H.

H can be written as $H = \{e^{i\theta} \,|\, 0 \leq \theta < 2\pi\}$ The cosets of H are the sets

$$rH = \{re^{i\theta} \,|\, 0 \leq \theta < 2\pi\}$$

This is the circle or radius r centered at the origin in the complex plane. For each $r > 0$ we get a different coset.

21. Suppose that G is an abelian group with an odd number of elements. Show that the product of all the elements of G is the identity.

Since the order of G is odd, the group does not contain any elements of order 2. (This uses the theorem proved in this chapter that the order of each element g divides the order of the group.) Therefore, $g \neq g^{-1}$ for all nontrivial elements $g \in G$. Since G is abelian we can multiply the elements in any order. So, we can multiply each element with its inverse and then there is e left over. The product is equal to e.

Another proof is the following. Let g be the product of all elements of G. Then g^{-1} is also the product of all the elements of G (in reverse order). So, $g = g^{-1}$ if G is abelian. So, $g^2 = e$ and the order of g is either 1 or 2. If the group has odd order, $|g|$ cannot be 2. So, $|g| = 1$ and $g = e$.

22. Suppose that G is a group with more than one element and G has no proper, nontrivial subgroups. Prove that $|G|$ is prime. (Do not assume at the onset that G is finite.)

Since G has more than one element, G contains the identity and at least one other element, say g. Look at the subgroup $\langle g \rangle$ of G generated

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by g. Since $g \neq e$ this subgroup is nontrivial. So, it cannot be proper. Therefore it must be the whole group: $\langle g \rangle = G$. Therefore G is cyclic.

If G is infinite, then $G \cong \mathbb{Z}$ and it has many nontrivial proper subgroups. So, G must be finite. The order of G must be prime. Otherwise $|G| = nm$ and g^n generates a subgroup of order m which is nontrivial and proper, giving a contradiction.

25. Let $|G| = 33$. What are the possible orders for the elements of G? Show that G must have an element of order 3.

Since the orders of the elements of G divide 33, the possible orders are $1, 3, 11, 33$.

Case 1. If G has an element g of order 33 then $\langle g \rangle = G$ and g^{11} has order 3.

Case 2. If G has no element of order 33 then I claim that G must contain an element of order 3. If it doesn't then all of the 32 nontrivial elements of G have order 11. For each $g \neq e$, $H = \langle g \rangle$ will be a cyclic subgroup of order 11.

Lemma 1. If H_1, H_2 are two cyclic subgroups of G of order 11 then either $H_1 = H_2$ or $H_1 \cap H_2 = \{e\}$.

Proof: $K = H_1 \cap H_2$ is a subgroup of H_1. Therefore the order of K divides $|H_1| = 11$. If $|K| = 11$ then $K = H_1 = H_2$. When these subgroups are different the intersection must be trivial: $|K| = 1$ and $K = \{e\}$. $\square$

Lemma 2. Any union of k distinct subgroups of G of order 11 will have $10k + 1$ elements.

Proof This is true for $k = 1$ suppose it is true for k. Thus the set

$$H_1 \cup H_2 \cup \cdots \cup H_k$$

has $10k + 1$ elements. Let H_{k+1} be another subgroup of G of order 11. Then H_{k+1} meets every other H_i in exactly one point, namely the identity e. Therefore, the 10 other elements of H_{k+1} are new, i.e., not contained in the union $H_1 \cup \cdots \cup H_k$. So, the new union

$$H_1 \cup H_2 \cup \cdots \cup H_k \cup H_{k+1}$$

will have exactly 10 more elements making $10k + 11 = 10(k + 1) + 1$. So, the formula holds for $k + 1$ and by induction it holds for all k. $\square$

Returning to our problem, the group G must be a union of subgroups of order 11 if it has no elements of order 3. Lemma 2 then says that $33 = 10k+1$ which is impossible. So, G must have at least one element of order 3.

30. Prove that every subgroup of D_n of odd order is cyclic.

This proof is a combination of the following theorems.

(1) If $g \in G$ then the order of g divides the order of G.
(2) The dihedral group has $2n$ elements consisting of n reflections and n rotations.
    (a) The reflections all have order 2 and
    (b) the rotations form a cyclic group of order n.
(3) Every subgroup of a cyclic group is cyclic.

Now suppose that H is a subgroup of D_n of odd order. Then H cannot contains any elements of order 2 by (1). Therefore, the elements of H cannot be reflections by (2a). So H is contained in the subgroup of D_n consisting of rotations. Since the rotation subgroup of D_n is cyclic (2b), H must be cyclic by (3).