MATH 30A, HOMEWORK 3

5. Homework 5

p. 23 #12, 16 and p. 165 #16, 20, 22.

0.12. Let a, b be positive integers and let $d = gcd(a, b)$ and $m = lcm(a, b)$ If t divides both a, b prove that t divides d. If s is a multiple of both a, b prove that s is a multiple of m.

By the Euclidean algorithm we have:

$$d = xa + yb$$

for some integers x, y. If t divides both a and b then $a = tv, b = tw$ and

$$d = t(xv + yw)$$

So, $t|d$.

Suppose that s is a multiple of a and b. We want to show that s is a multiple of m. So, divide s by m and get a remainder r where $0 \leq r < m$:

$$s = qm + r, \quad 0 \leq r < m$$

Since a and b divide s and they divide m, they also divide $r = s - qm$. So, r is a common multiple of a and b. But m is the smallest positive common multiple of a and b. This gives a contradiction unless $r = 0$. So, we must have $r = 0$ and m divides s.

0.16. Use the Euclidean algorithm to find $gcd(34, 126)$ and write it as a linear combination of $34, 126$.

x	y	$ax + by$	*quotient*
1	0	$a = 126$	
0	1	$b = 34$	3
1	-3	24	1
-1	4	10	2
3	-11	4	2
-7	26	2	

The result is that $2 = -7(126) + 26(34)$.

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8.16. Determine the number of elements of order 15 and the number of cyclic subgroups of order 15 in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$.

$\mathbb{Z}_{30}$ has 2 elements of order 3 (namely, 10 and 20) and 8 elements of order 15 (namely, 2,4,8,14,16,22,26,28). $\mathbb{Z}_{20}$ has 4 elements of order 5 (If p divides n then $\mathbb{Z}_n$ has exactly $p-1$ elements of order p. They are the multiples of n/p.)

There are $8 + 40 = 48$ elements of order 15. There are 8 elements of the form $(10, y)$ or $(20, y)$ where $y \in \mathbb{Z}_{20}$ has order 5 and $8 \cdot 5 = 40$ more of the form (x, y) where $x \in \mathbb{Z}_{30}$ has order 15 and $y \in 6\mathbb{Z}_{20}$..

There must be exactly 48/8=6 cyclic subgroups of order 15. The reason is that each one has 8 elements of order 15 and these have to be disjoint sets. If two cyclic groups of order 15 share an element of order 15 they must be equal.

8.20. The group $S_3 \oplus \mathbb{Z}_2$ is isomorphic to one of the following groups (list of all 4 groups of order 12): $\mathbb{Z}_{12}, \mathbb{Z}_6 \oplus \mathbb{Z}_2, A_4, D_6$. Determine which one by elimination.

Since S_3 is not abelian, neither is $S_3 \oplus \mathbb{Z}_2$. So, this group cannot be isomorphic to either of the abelian groups $\mathbb{Z}_1 2, \mathbb{Z}_6 \oplus \mathbb{Z}_2$. If we count the number of elements of order $2, 3$ and 6 we will see that $S_3 \oplus \mathbb{Z}_2$ cannot be isomorphic to A_4.

group	order 1	order 2	order 3	order 6
A_4	1	3	8	0
D_6	1	7	2	2
$S_3 \oplus \mathbb{Z}_2$	1	7	2	2

8.22. Find a subgroup of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ that is not of the form $H \oplus K$ where $H \leq \mathbb{Z}_4, K \leq \mathbb{Z}_2$.

There are two cyclic subgroups which are not of the form $H \oplus K$.

(1) $\langle(1, 1)\rangle = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$ If this had the form $H \oplus K$ then H would be the set of all first coordinates $H = \{0, 1, 2, 3\} = \mathbb{Z}_4$ and K would have to be the set of all second coordinates $K = \{0, 1\} = \mathbb{Z}_2$ which appear in this subgroup. But $H \oplus K$ has 8 elements and $\langle(1, 1)\rangle$ has only 4, so they are not equal.

(2) $\langle(2, 1)\rangle = \{(0, 0), (2, 1)\}$ does not contain $(0, 1)$, so it is not of the form $H \oplus K$.