MATH 30A, HOMEWORK 6

6. Homework 6: factor groups

p. 191 #9.12, 16, 21, 24, 30

8.12. Prove that a factor group of an abelian group is abelian.

Suppose that G is abelian and H is a subgroup of G. Then for any two elements aH, bH of G/H we have:

$$(aH)(bH) = abH = baH = (bH)(aH)$$

8.16. Recall that $Z(D_6) = \{e, R_\pi\}$. What is the order of $R_{\pi/3}Z(D_6)$ in $D_6/Z(D_6)$?

In general, the order of aH in G/H is the smallest positive integer n so that $a^n \in H$. In this case $R_{\pi/3}^n = R_{n\pi/3}$ is in $Z(D_6)$ if and only if n is a multiple of 3. So the smallest positive n is 3. So, the order of $R_{\pi/3}Z(D_6)$ is 3.

8.21. Prove that an abelian group of order 33 is cyclic.

This is pretty easy if we use the classification of finite abelian groups. But maybe we aren't supposed to do that because that comes later in the book!

What is proved in this chapter is that, if p divides the order of an abelian group, then the group has an element of order p. So, this group, call it G has an element of order 3 (say a) and an element of order 11 (say b). Then ab has order 33 since $(ab)^{33} = a^{33}b^{33} = e$ but $(ab)^3 = a^3b^3 = b^3 \neq e$ and $(ab)^{11} = a^{11}b^{11} = a^2 \neq e$. Then $G = \langle ab \rangle$ is the cyclic group generated by ab.

8.24. Identify the group $(\mathbb{Z}/4 \oplus \mathbb{Z}/12)/\langle (2,2) \rangle$

Since $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$ has 48 elements and

$$H := \langle (2,2) \rangle = \{(0,0), (2,2), (0,4), (2,6), (0,8), (2,10)\}$$

has 6 elements, the quotient group has $48/6 = 8$ elements. But $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$ has no elements of order 8 since 12 times any element is zero.

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This makes the orders of all elements divisors of 12. So, the orders of the elements of the factor group are also divisors of 12.

The factor group also has an element of order 4, namely $(1,0)+H$. The order of this element is the smallest positive integer n so that $n(1,0)=(n,0)$ is an element of H. Looking at the list of elements, we see that there is no nontrivial element in H with second coordinate 0. So, $(n,0)=(0,0)$, i.e., n is divisible by 4. So, the order of $(1,0)+H$ is 4.

Since G/H has no elements of order 8, it is not isomorphic to $\mathbb{Z}_8$. Since G/H has at least one element of order 4, G/H is not isomorphic to $\mathbb{Z}_2^3=\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$. So, the only remaining possibility is

$$G/H \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2.$$

8.30. Express $U(165)$ as an internal direct product of proper subgroups in 4 different ways.

Since $165 = 3\cdot 5\cdot 11$ we have

(1) $U(165)=U_3(165)\times U_{55}(165)$
(2) $U(165)=U_5(165)\times U_{33}(165)$
(3) $U(165)=U_{11}(165)\times U_{15}(165)$
(4) $U(165)=U_{55}(165)\times U_{33}(165)\times U_{15}(165)$

where

$$\begin{aligned}
U_{55}(165) = &\ \{x\in U(165)\,|\,x\equiv 1 \mod 55\} = \{1,56\}\\
U_{33}(165) = &\ \{1,34,67,133\}\\
U_{15}(165) = &\ \{1,16,31,46,61,76,91,106,136,151\}\\
U_{11}(165) = &\ \{1,23,34,56,67,89,122,133\}\\
U_{5}(165) = &\ \{1,16,26,31,41,46,56,61,71,76,86,91,\\
&\ 101,106,116,131,136,146,151,161\}\\
U_{3}(165) = &\ \{1,4,7,13,16,19,28,31,34,37,43,46,49,52,58,61,64,67,73,76,79,82,91,\\
&\ 94,97,103,106,109,112,118,124,127,133,136,139,142,148,151,157,163\}
\end{aligned}$$