# MATH 30A, HOMEWORK 7, CHAP 10

## 10. Homomorphisms

10.1. What are the possible kernels of the homomorphisms from $\mathbb{Z}$ to $S_5$? Find a homomorphism from $\mathbb{Z}$ to $S_5$ with kernel $6\mathbb{Z}$.

3) The hints: Can you describe all the homomorphisms from $\mathbb{Z}$ into an arbitrary group $G$? What is the kernel of each of these homomorphisms? (Remember that $\mathbb{Z}$ is additive and $G$ is multiplicative. The exponential function $f(n) = e^n$ is an example where $G = \mathbb{R}^*$) (For some reason it is easier to do an arbitrary group than a specific group.)

Homomorphisms $\phi : \mathbb{Z} \to G$ satisfy the equation $f(n) = f(1)^n$. So they are determined by $a = \phi(1)$. Given any $a \in G$ the corresponding homomorphism $\phi : \mathbb{Z} \to G$ is given by

$$\phi(k) = a^k$$

which can be an arbitrary element of the group. The kernel of $\phi$ is $n\mathbb{Z}$ where $n$ is the order of $a = \phi(1)$.

4) Now do it for $G = S_5$.

The orders of the elements of $S_5$ are: $1, 2, 3, 4, 5, 6$. So, the possible kernels of homomorphisms $\phi : \mathbb{Z} \to S_5$ are: $\mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}, 4\mathbb{Z}, 5\mathbb{Z}, 6\mathbb{Z}$. To find a homomorphism with kernel $6\mathbb{Z}$ we need an element of $S_5$ with order 6. This could be $(12)(345)$. So, a homomorphism $\phi : \mathbb{Z} \to S_5$ with kernel $6\mathbb{Z}$ is given by

$$\phi(n) = (12)^n(345)^n$$

10.2. Find all homomorphisms from $S_5$ to $\mathbb{Z}$.

Again, look at the hints:

(1) What are the finite subgroups of $\mathbb{Z}$? Only the trivial subgroup $\{0\}$ is finite.
(2) What does that tell you about homomorphisms from a finite group into $\mathbb{Z}$? The image of any homomorphism from a finite group $G$ into $\mathbb{Z}$ is a finite subgroup of $\mathbb{Z}$ and therefore trivial.

*Date*: November 6, 2006.

So, $\phi$ must be trivial. The only homomorphism $\phi : S_5 \to \mathbb{Z}$ is the trivial one. ($\phi(x) = 0$ for all $x \in S_5$)

10.3. If $N$ is a cyclic normal subgroup of $G$ of order 100 and index 101 then show that $G$ is cyclic.

Let's follow the outline given in the hints:

a) Show $\mathrm{Aut}(N)$ has no elements of order 101.

Since $N \cong \mathbb{Z}_{100}$ (Every cyclic group of order $n$ is isomorphic to $\mathbb{Z}_n$.), $\mathrm{Aut}(N)$ is isomorphic to $U(100)$ which has

$$n \prod (1 - 1/p) = 100(1 - 1/2)(1 - 1/5) = 100 \left(\frac{1}{2}\right)\left(\frac{4}{5}\right) = 40$$

elements. This is not divisible by 101. So, $\mathrm{Aut}(N)$ cannot have an element of order 101.

b) Therefore $N$ is central.

The $N/C$ theorem says that the number $|N_G(N) : C(N)|$ must divide the order of the $\mathrm{Aut}(G)$. Since this is not possible we have

$$N_G(N) = C_N(N)$$

But $N$ is normal. So, $G = N_G(N) = C_G(N)$. I.e., $N$ is central in $G$.

c) Use the theorem that if the index of the center is prime, then the group is abelian. OK, now we know that $G$ is abelian.

d) Use the theorem that says $G$ must have an element of order 101. Since 101 is a prime number dividing $|G|$, $G$ has an element, say $a$, of order 101.

e) Find a generator for $G$ to show it is cyclic.

Let $b$ be a generator of $N$. Then $b$ has order 100 and $ab$ has order 10100. Therefore, $G = \langle ab \rangle$ is cyclic.

Some students found another proof using Sylow and the following theorem:

**Theorem 10.1** (internal direct product)**.** *Suppose that $N, K$ are normal subgroups of $G$ so that*

(1) $N \cap K = \{e\}$
(2) $NK = G$. *(This is the same as $|N| \cdot |H| = |G|$ given (1).)*

*Then $G \cong N \oplus K$.*

We already have the normal subgroup $N \cong \mathbb{Z}_{100}$. By Cauchy's thm we have an element $b \in G$ of order 101. Let $K = \langle b \rangle$. Then $N \cap K = \{e\}$ since $|N \cap K|$ is a number that divides both $|N| = 100$ and $|K| = 101$.

Now we use the Sylow thms to tell us that $K$ is normal in $G$. One way is using the 3rd Sylow thm which implies that the index of the normalizer of $K$ is congruent to 1 modulo 101. But this index is $|G : N(K)| \leq 100$. So, it must be 1.

Another way is to say that if $K$ is not normal then it is not equal to one of its conjugates, say $H = gKg^{-1}$. If $K \neq H$ we have $K \cap H = \{e\}$ (since it is a proper subgroup of each). But then $|HK| = |H| \cdot |K|/|H \cap K| = 101^2$ which is impossible. So, $K$ is the only conjugate of $K$. So, $K \triangleleft G$. So the above theorem says

$$G \cong N \oplus K \cong \mathbb{Z}_{100} \oplus \mathbb{Z}_{101} \cong \mathbb{Z}_{10100}$$

since 100 and 101 are coprime.

10.4. Suppose that $G$ is a group and $A$ is an abelian group. Let $f, g : G \to A$ be two homomorphism. Show that the set $S = \{x \in G \mid f(x) = g(x)\}$ is a normal subgroup of $G$.

Since $A$ is abelian, $\phi(x) = f(x)g(x)^{-1}$ is a homomorphism from $G$ to $A$:

$$\begin{aligned} \phi(xy) &= f(xy)g(xy)^{-1} = f(x)f(y)g(y)^{-1}g(x)^{-1} \\ &= f(x)g(x)^{-1}f(y)g(y)^{-1} = \phi(x)\phi(y) \end{aligned}$$

$S$ is the kernel of $\phi$ and is therefore a normal subgroup of $G$.

What happens if $A$ is not abelian?

In that case, $S$ is a subgroup of $G$ which may not be normal. For example, if $G = A = S_3$, $f(x) = x$ and $g(x) = (12)x(12)$. Then $S = C((12)) = \{e, (12)\}$ which is not normal in $S_3$.