MATH 30A: QUIZZES

Answers to Practice Quiz I

1. Find an example of a group with 8 elements which is not cyclic.

Here are two examples that people came up with: The dihedral group D_4 has 8 elements and is not cyclic because it is nonabelian. (Actually the question does not ask for a proof or explanation. But if it asked this would be enough.) The automorphism group of $\mathbb{Z}_{16}$ also has 8 elements and is not cyclic since it has 3 elements of order 2, namely, $\psi_7, \psi_9, \psi_{15}$. However, it was not my intension that you need to use things you just learned.

2. Find a specific example of a group G which is nonabelian and a subgroup H which is abelian. (Abelian is the same as commutative.)

The simplest example is $G = S_3$ and $H = \{e\}$. But another popular example was $G = GL(2, \mathbb{R})$ and $H =$ the group of diagonal matrices with nonzero diagonal entries

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \neq 0 \right\}$$

The dihedral group $G = D_n$ is another example with $H = \mathbb{Z}_n$ or $H = \langle r \rangle$ where r is a reflection.

3. If $g \in G$ has order 6 then show that g^2 has order 3. Give an example where $G = S_6$.

This seem to cause some confusion. I meant take the group S_6 and give me one specific element $g \in S_6$ which has order 6 and show that, for that particular element, g^2 has order 3.

The general proof is as follows. Since g has order 6, $(g^2)^3 = g^6 = e$, $g^2 \neq e$ and $(g^2)^2 = g^4 \neq e$. The first equation proves that the order of g is 3 or less. The second and third equation prove that the order of g^2 cannot be 1 or 2. Therefore, $|g^2| = 3$.

An example is: $g = (123456)$ which has order 6 and $g^2 = (135)(246)$ which has order $lcm(3, 3) = 3$.

Date: September 28, 2006.

4. Verify that the following is a group. X is a set with 1,000,000 elements. G is the set of all subsets of X. (So, G has $2^{1000000}$ elements. You don't have to prove that.) The binary operation on G is symmetric difference:

$$A \oplus B = (A - B) \cup (B - A)$$

This is the set of all elements in either A or B but not both. You may use Venn diagrams to prove associativity.

Since $A \oplus B$ is a subset of X, it is an element of G. Therefore, the set G is closed under $\oplus$. There are 3 more things to verify:

(1) The group has an identity: This is the empty set $\emptyset$.
(2) The group has inverses: Every set is its own inverse $A \oplus A = \emptyset$.
(3) The operation is associative: Draw the Venn diagram. $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ is the region which looks like a shaded triangle with three giant blades attached at the corners.