MATH 30A: QUIZZES

2. Answers to Quiz 2 review

Quiz 2 is on Thursday, Oct 5. It will have 4 questions. Open book and notes. You may also bring a calculator and/or laptop computer. But no fair using internet or IM with friends!

2.1. **isomorphisms.**

2.1.1. Find an example of two infinite groups with the same number of elements which are not isomorphic and give a reason.

There are lots of examples.

(1) $GL(2,\mathbb{Z})$ and $\mathbb{Z}$ are both countably infinite but they are not isomorphic because the first is nonabelian and the second is abelian.
(2) Z and $\mathbb{Q}$ are both countably infinite but the first is cyclic and the second is not so they cannot be isomorphic. To see that $\mathbb{Q}$ is not cyclic, suppose that it is. Then there is a rational number a/b so that all other rational numbers are integer multiples. I.e., all rational numbers must have the form na/b. However, $1/2b$ is not of this form, a contradiction.
(3) $(\mathbb{R}_{\neq 0},\cdot)$ and $(\mathbb{R},+)$ have the same cardinality but they are not isomorphic because the first group has an element of order 2 (namely, -1) whereas the second does not.

2.1.2. Write down an explicit isomorphism between the groups $G = (\mathbb{R},+)$ and $H =$ the subgroup of $GL(2,\mathbb{R})$ given by

$$H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

This is easy. The isomorphism $\phi : G \to H$ is given by

$$\phi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

This is obviously a bijection. To see that it is an isomorphism, you just compute:

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$$\phi(x)\phi(y) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \cdot y + x \cdot 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = \phi(x+y)$$

2.1.3. Find an example of an automorphism ϕ of a nontrivial group G so that $\phi(x) \neq x$ for all nontrivial ($x \neq e$) elements of G.

The example that I was thinking of was for a cyclic group $G = \mathbb{Z}_n$ of odd order. So $n = 2k+1$. Then the mapping $\phi(x) = -n$ (modulo n) is an automorphism which does not fix any element except 0.

2.1.4. Find as many (nonisomorphic) examples as you can of groups G so that $\text{Aut}(G)$ has only one element.

The "obvious" examples are $G = \{e\}$ (the trivial group) and $G = \mathbb{Z}_2$. I think these are the only examples but I couldn't give a complete proof.

2.2. permutations.

2.2.1. Find an element of S_{12} of order 60.

$(1234)(567)(8, 9, 10, 11, 12)$ has order $4 \cdot 3 \cdot 5 = 60$.

2.2.2. Show that all permutations of odd order are even.

Every odd permutation is a product of odd cycles. Every odd cycle is an even permutation and the product of even permutations is even.

2.2.3. Prove the following formula:

$$\sigma(123 \cdots n)\sigma^{-1} = (\sigma(1)\sigma(2)\sigma(3) \cdots \sigma(n))$$

Let τ denote this permutation. If $x = \sigma(i)$ where $1 \leq i < n$ then

$$\tau(\sigma(i)) = \sigma(12 \cdots n)\sigma^{-1}(\sigma(i)) = \sigma(12 \cdots n)(i) = \sigma(i+1)$$

and

$$\tau(\sigma(n)) = \sigma(12 \cdots n)\sigma^{-1}(\sigma(n)) = \sigma(12 \cdots n)(n) = \sigma(1)$$

For other letter $x = \sigma(k)$ with $k > n$ we have

$$\tau(x) = \tau(\sigma(k)) = \sigma(12 \cdots n)\sigma^{-1}(\sigma(k)) = \sigma(12 \cdots n)(k) = \sigma(k) = x$$

So, $\tau = (\sigma(1)\sigma(2) \cdots \sigma(n))$.

2.3. cyclic groups.

2.3.1. Find as many (nonisomorphic) examples as you can of cyclic groups having only two generators.

The examples that I thought of were $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$.

2.3.2. Suppose that a, b are elements of a group G and $ab = ba$. Suppose that $|a| = 8, |b| = 30$. What are the possible values of $|ab|$?

Since $ab = ba$ (let's call this $c = ab = ba$) we know that $c^n = a^n b^n$. Therefore, $c^{120} = a^{120}b^{120} = e$. So, the order of c is a divisor of $lcm(8, 30) = 120$. Now suppose that $n = |c|$. Then, by the same argument, $|a| = 8$ divides $lcm(n, 30)$. This means that n is a multiple of 8. And $|b| = 30$ divides $lcm(n, 8)$. This means that n is a multiple of 15. So, n is a multiple of 120. So, $n = 120$.

2.3.3. Show that aba^{-1} has the same order as b.

Let $c = aba^{-1}$. Then

$$c^n = aba^{-1}aba^{-1} \cdots aba^{-1} = ab^n a^{-1}$$

(and $c^{-n} = (c^n)^{-1} = (ab^n a^{-1})^{-1} = ab^{-n}a^{-1}$.). So, $c^{|b|} = aea^{-1} = e$. So, $|c|$ divides $|b|$. But $b = xcx^{-1}$ where $x = a^{-1}$. So, the same argument shows that $|b|$ divides $|c|$ so $|b| = |c| = |aba^{-1}|$.

2.4. subgroups.

2.4.1. How many subgroups does D_4 have? List them. (You don't have to list the elements of each.)

(Actually, this is unfair. It is very difficult to find the two subgroups H_2, H_3 which are not cyclic.) But, here is the answer: Write the elements of D_4 as $D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$. Then there are 10 subgroups. 7 are cyclic. 3 are not.

(1) One trivial subgroup: $\{e\}$.
(2) Five of order 2: $\langle r^2\rangle, \langle s\rangle, \langle sr\rangle, \langle sr^2\rangle, \langle sr^3\rangle$
(3) Three of order 4: $H_1 = \langle r\rangle, H_2 = \{e, s, r^2, sr^2\}, H_3 = \{e, r^2, sr, sr^3\}$
(4) The whole group D_4.

2.4.2. Suppose that H, K are subgroups of a group G which centralize each other in the sense that $hk = kh$ for all $h \in H, k \in K$. Then show that the set

$$HK := \{hk \,|\, h \in H, k \in K\}$$

is a subgroup of G.

You just verify the definition:

(1) e is in the set HK since $e = ee$.
(2) If $hk \in HK$ then $(hk)^{-1} = k^{-1}h^{-1} = h^{-1}k^{-1} \in HK$
(3) If $h_1k_1, h_2k_2 \in HK$ then their product

$$h_1k_1h_2k_2 = h_1h_2k_1k_2 \in HK$$

This is related to the previous question. $H = \langle s\rangle$ and $K = \langle r^2\rangle$ centralize each other and $HK = H_2$. Similarly, $H_3 = \langle sr\rangle K$.

2.4.3. The group $\mathbb{Z}_8$ has eight elements and only one subgroup of order 4. The group D_4 has order 8 and 3 subgroups of order 4 (check your answer to 2.4.1). Can you find a group of order 8 which has more than 3 subgroups of order 4? [Hint: look at Practice quiz 1.4 where X has 3 elements.]

Let $X = \{1, 2, 3\}$ and let G be the group of all subsets of X under $\oplus$. Writing this multiplicatively the group is:

$$G = \{e, a, b, c, ab, bc, ac, abc\}$$

where $a = \{1\}, b = \{2\}, c = \{3\}$. $ab = a \oplus b = \{1, 2\}$, etc. The subgroups of G of order 4 are

(1) $H_1 = \{e, a, b, ab\}$
(2) $H_2 = \{e, a, c, ac\}$
(3) $H_3 = \{e, b, c, bc\}$
(4) $H_4 = \{e, a, bc, abc\}$
(5) $H_5 = \{e, ab, c, abc\}$
(6) $H_6 = \{e, b, ac, abc\}$
(7) $H_7 = \{e, ab, bc, ac\}$

(The question asks for "more than 3." So, you can stop after you find 4.)