MATH 30A: QUIZZES

3. Review for Quiz 3

Quiz 3 is on Thursday, Oct 19. It will have 1 question. Open book and notes. You may also bring a calculator and/or laptop computer. But no internet or IM with friends. No cellphone use. The quiz will be on cosets and direct products.

3.1. **Cosets.**

3.1.1. List all the left cosets of $H = \langle r^2 \rangle$ in the group D_{2n} which consists of the rotations $e, r, r^2, \cdots, r^{2n-1}$ where $r = R_{\pi/n}$ and the reflections $s, sr, sr^2, \cdots, sr^{2n-1}$. Find all the right cosets.

The index of H in D_{2n} is $|D_{2n}|/H = 4n/n = 4$. So there are exactly 4 left cosets. They are

$$H = \{e, r^2, r^4, \cdots\}$$

$$rH = \{r, r^3, r^5, \cdots\}$$

$$sH = \{s, sr^2, sr^4, \cdots\}$$

$$srH = \{sr, sr^3, sr^5, \cdots\}$$

Since r has order $2n$, $sr = r^{2n-1}s$. So, the right cosets are the same as the left cosets: $Hr = rH, Hs = sH, Hsr = srH$. (Oops, I didn't intend to pick a normal subgroups! In general the left and right cosets are different.)

3.1.2. Prove that Ha is the set of all inverses of elements of $a^{-1}H$.

$a^{-1}H$ is the set of all $a^{-1}h$ where $h \in H$. The inverses of these elements are $h^{-1}a$. But, as h runs over all the elements of H, so does h^{-1}. So,

$$\{h^{-1}a \,|\, h \in H\} = \{ha \,|\, h \in H\} = Ha$$

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3.1.3. Let $G = S_6$ (the symmetric group on 6 letters). $H = \langle(123)\rangle$. Find two elements $a, b \in S_6$ so that $aH = bH$ but $Ha \neq Hb$.

You need to remember that $aH = bH$ if and only if $a^{-1}b \in H$ and $Ha = Hb$ iff $ba^{-1} \in H$. So, start with $(123) \in H$. I want this to be $a^{-1}b$. Take $a = (145)$ Then $b = (12345) = a(123)$. So, $aH = bH$. But

$$ba^{-1} = (12345)(154) = (15)(234) \notin H$$

So $Ha \neq Hb$.

3.1.4. Suppose that H, K are subgroups of G and $H \subseteq K$. Then show that every left coset of H is contained in a left coset of K. State and prove the converse.

Every left coset of H is contained in a left coset of K: Take any left coset aH. Then $H \subseteq K$ implies $aH \subseteq aK$.

The converse is: If every left coset of H is contained in a left coset of K then $H \subseteq K$. To prove this note that $H = eH$ is a left coset of H. So, H is contained in some left coset aK of K. But $e \in H \subseteq aK$ implies $e \in aK$. So, $aK = eK = K$. Therefore, $H \subseteq K$ as claimed.

3.2. **Direct products.**

3.2.1. If $G \oplus H$ is abelian then prove that G and H are abelian.

To show that G is abelian, I need to show that $ab = ba$ for all $a, b \in G$. I know that $G \oplus H$ is abelian. So,

$$(a, e)(b, e) = (b, e)(a, e)$$

Multiplying gives $(ab, e) = (ba, e)$. This means $ab = ba$. So, G is commutative. The argument for H is similar.

3.2.2. Find all groups G, H so that $G \oplus H$ is cyclic.

If $G \oplus H$ is cyclic then G, H must both be cyclic. And their orders, if finite, must be relatively prime. Conversely, the direct product of finite cyclic groups of relatively prime orders is cyclic. So, these are all the cases where $G \oplus H$ is cyclic and finite. For $G \oplus H$ to be cyclic and infinite, One of the groups must be infinite cyclic (isomorphic to $\mathbb{Z}$) and the other must be trivial.

3.2.3.

(1) If $a \in G, b \in H$ have order $|a| = 4, |b| = 6$, what is the order of (a, b) in the product group $G \oplus H$?
The order of (a, b) is $12 = lcm(4, 6)$.

(2) (a, b) (as in (1)) generates a subgroup $\langle(a, b)\rangle$ of $G \oplus H$ which is not of the form $A \oplus B$. Why not?
In a product group $A \oplus B$, the elements have the form (x, y) where $x \in A$ and $y \in B$. This implies that, whenever $(x, y) \in A \oplus B$, so are (a, e) and (e, b). The group $\langle(a, b)\rangle$ contains (a, b) but it does not contain (a, e) (in order to get the first coordinate equal to a, we need to raise to a power (< 12) congruent to 1 modulo 4, i.e., to power 1,5, or 9. But $(a, b)^1 = (a, b), (a, b)^5 = (a, b^5), (a, b)^9 = (a, b^3)$. None of these is (a, e).

(3) Complete the sentence: If $a \in G$ and $b \in H$ then the subgroup $\langle(a, b)\rangle$ of $G \oplus H$ is of the form $A \oplus B$ if ....
if a, b have finite, relatively prime orders, or if either a or b is e.

3.2.4. If H, aH, bH are the left cosets of H in G_1 and K, cK are the left cosets of K in G_2 what are the left cosets of $H \oplus K$ in $G_1 \oplus G_2$?
They are:

$$H \oplus K, (a, e)H \oplus K, (b, e)H \oplus K, (e, c)H \oplus K, (a, c)H \oplus K, (b, c)H \oplus K$$

Here, e.g., $(b, c)H \oplus K = bH \times cK = \{(bh, ck) \mid h \in H, k \in K\}$

What are the right cosets of $H \oplus K$ in $G_1 \oplus G_2$?
a,b might lie in the same right coset of H. But a^{-1}, b^{-1} lie in different right cosets. (Why?) So the right cosets of $H \oplus K$ are

$$H \oplus K, H \oplus K(a^{-1}, e), H \oplus K(b^{-1}, e), H \oplus K(e, c), H \oplus K(a^{-1}, c), H \oplus K(b^{-1}, c)$$