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The median grade was 16/30. This was the cutoff between A-/B+ for this quiz.
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4. Quiz 4

Rules: Choose two out of 3 numbered questions. Or do all three and I will take the best two. For example, if you get 5,3,2 your total is 8. The hints are not mandatory. You can solve the problems using any method.

(1) Suppose that G is a group of order $|G| = 351 = 3^3 13$ and P is a Sylow 13-subgroup.
   (a) If P is not normal in G then how many conjugates does P have in G? (Use Sylow's third.)
   The only divisors of 351 which are congruent to 1 modulo 13 are 1 and 27. If the number is 1 then P is normal. If P is not normal then the number of Sylow p-subgroups must be 27.
   (b) If H is a subgroup of G of order 39 then how many of the Sylow 13-subgroups lie in H?
   The only divisors of 39 which is congruent to 1 mod 13 is 1. Therefore, H has only one Sylow 13-subgroup, call it Q. Since the Sylow 13-subgroups of G which lie in H are Sylow 13-subgroups of H we concluded that H contains exactly one of the conjugates of P.
   (c) Suppose that H is a normal subgroup of G of index 3. Then show that P is normal in H and in G. (The second statement follows from the first. Why? Since it usually is not the case that a normal subgroup of a normal subgroup is normal, why should it be true in this case?)
   By the same argument as above, H has a unique Sylow 13-subgroup. Call it Q. Since $|Q| = 13$, Q is also a Sylow 13-subgroup of G. Therefore, by Sylow, $P = gQg^{-1}$ for some $g \in G$. But
   $$Q \le H \Rightarrow P = gQg^{-1} \le gHg^{-1} = H$$
   since $H \triangleleft G$. This implies that $Q = P$ since H has only one Sylow 13-subgroup. This argument show that $gQg^{-1} = Q$ for all $g \in G$. Therefore, $Q = P$ is normal in G.
   Note: If $K \triangleleft H$ and $H \triangleleft G$ then K is called a *subnormal* subgroup of G since it may not be normal.

(2) Suppose that $\phi : G \to H$ is an epimorphism (a homomorphism which is onto). Suppose that the kernel P of ϕ is a Sylow p-subgroup of G. Then for any subgroup K of G show that $K \cap P$ is the Sylow p-subgroup of K (assuming that it is nontrivial). Give an example. Hints:
   (a) Does p divide the order of H?
   No. Suppose that $|G| = p^k m$. Then $H \cong G/P$ has order m which is prime to p.
   (b) $K/K \cap P \cong \phi(K)$. Why is that?
   The restriction of ϕ to K gives a homomorphism $\phi|K : K \to H$. The kernel of $\phi|K$ is $K \cap \ker \phi = K \cap P$. The first isomorphism theorem says that the image of K is isomorphic to $K/\ker \phi|K = \frac{K}{K \cap P}$.
   (c) How do you [know] that p does not divide the order of $\phi(K)$?
   Since $\phi(K)$ is a subgroup of H, the order of $\phi(K)$ divides the order of H. So, $|\phi(K)|$ cannot be a multiple of p.

(d) How do you know that $|K \cap P|$ is a power of p?
This is a subgroup of P so its order divides $|P| = p^k$. So, it must be a power of p.

(e) Conclude that $K \cap P$ is the Sylow p-subgroup of K.
It is a p-subgroup of K whose index is prime to p. So, it must be a Sylow p-subgroup. Several of you pointed out that, by Sylow's 2nd theorem, any Sylow p-subgroup of K must be contained in P and therefore in $K \cap P$. So, $K \cap P$ is a p-subgroup of K which contains every Sylow p-subgroup of K. So, it must itself be a Sylow p-subgroup of K.

(f) For an example, try $G = D_9$ with $H = \mathbb{Z}_2$ and $K = D_3$. You could also use an abelian example with $G = \mathbb{Z}_6 \oplus \mathbb{Z}_6$, $H = 2\mathbb{Z}_6 \oplus 2\mathbb{Z}_6$ and $K = \mathbb{Z}_6 \oplus 0$.
Let $\phi : D_9 \to \mathbb{Z}_2$ be the homomorphism which takes all reflections sr^i to 1 and all rotations r^i to 0. The rotations form a cyclic group $\langle r \rangle$ of order 9. This is the unique Sylow 3-subgroup of D_9. Then $K \cap \langle r \rangle = \langle r^3 \rangle$ is the Sylow 3-subgroup of $K = D_3$.
In the abelian case, $\phi : \mathbb{Z}_6 \oplus \mathbb{Z}_6 \to H = 2\mathbb{Z}_6 \oplus 2\mathbb{Z}_6$ is multiplication by 2. The kernel is the subgroup with 4 elements: $P = \{(0,0), (3,0), (0,3), (3,3)\}$. This is the Sylow 2-subgroup of $\mathbb{Z}_6 \oplus \mathbb{Z}_6$. The image of $K = \mathbb{Z}_6 \oplus 0$ is $2\mathbb{Z}_6 \oplus 0$ which is cyclic of order 3. So, the image is isomorphic to K modulo $K \cap P = \{(0,0), (3,0)\}$.

(3) Show that any group of order $2405 = 5 \cdot 13 \cdot 37$ is cyclic. Hints:

(a) Let a, b, c be elements of G of order $5, 13, 37$.
These exist by Cauchy's theorem.

(b) Use Sylow theorems to prove that $P_{37} = \langle c \rangle$ is normal.
The index of P_{37} is either 1,5,13 or 65. But only 1 is congruent to 1 mod 37. So, there is only one Sylow 37-subgroup. So, it must be normal

(c) Use N/C theorem to prove that P_{37} is central.
This theorem says that $N(H)/C(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$. In this case, $N(H) = G$ and $C(H)$ contains $H = P_{37}$. So, the index of $C(P_{37})$ in G is either 1,5,13 or 65. But $\text{Aut}(\mathbb{Z}_{37}) = U(37)$ has 36 elements and 5,13 do not divide 36. Therefore, $G/C(P_{37})$ has order 1. So, $C(P_{37}) = G$ which means that P_{37} is central. This implies that c is central.

(d) Do the same thing for the Sylow 13-subgroup.
By the same argument P_{13} is central in G. So, b is also central.

(e) Show that the product abc has order 2405.
Since b, c are central, $ab = ba$, $bc = cb$ and $ac = ca$. I.e., the elements a, b, c commute with each other. Therefore, $(abc)^n = a^n b^n c^n$. (Otherwise we would have $(abc)^n = abcabcabcabc \cdots abc$.)
Since $5, 13, 37$ are relatively prime, we can now conclude that abc has order $5 \cdot 13 \cdot 37 = 2405$. So, G is cyclic.