5. Review for Quiz 4

Quiz 4 is on Thursday, Nov 16. It will have 2 questions. Open book and notes. You may also bring a calculator and/or laptop computer.

5.1. **First Isomorphism Theorem.**

5.1.1. *statement.* What does the first isomorphism theorem say? Give an example using a homomorphism which is neither onto or 1-1 *and nontrivial* . In particular, find such a homomorphism from D_4 to S_3.

We did this in class. And I added the word *nontrivial* because there is always the trivial homomorphism.

We constructed the homomorphism in the following way. First, we decided to make $\ker\phi$ equal to the cyclic group of all rotations: $K = \langle r\rangle$ Then we decided to make the image equal to the group $\{e, (12)\}$. Then we constructed the only mapping that ϕ could be.

5.1.2. Suppose that G is a finite group and $\phi : G \to H$ is a homomorphism. Then show that $\phi(P)$ is a Sylow p-subgroup of $\phi(H)$ for any Sylow p-subgroup of G. (Dumb=too simple question.) Also not quite true. What is the correct statement? (You need to exclude the case when $\phi(P)$ is too small.

5.2. **Sylow theorems.**

5.2.1. If G is a group of order pq where $p < q$ then show that either the Sylow p-subgroup of G or the Sylow q-subgroup of G is normal. (Also dumb) The Sylow q-subgroup must be normal.

How many elements of order p are there? Hint: Every cyclic group of order p has $p-1$ elements of order p (and one element of order 1).

5.2.2. If $|G| = pqr$ where $p < q < r$ are prime. Take $p = 5, q = 7, r = 11$, what can you say? Which Sylow subgroups are normal? About notation: If you don't know the standard notation you can make something up but you need to explain your notation. For example "I will use P_p to denote a Sylow p-Subgroup. If there is more than one Sylow p-subgroup then I will call them P_p^1, P_p^2, etc."

So, for this problem I could write: By the Sylow theorems there is a unique Sylow 7-subgroup P_7 and a unique Sylow 11-subgroup P_{11} and there might be several Sylow 5-subgroups: call them Q_1, Q_2, etc.

How many are there when they are not normal? The number of Sylow 5-subgroups is either 1 or 11. Can you do the next step in the analysis of this problem: There are only two groups G of order 385. One is the cyclic group. The other is a group which has 11 Sylow 5-subgroups.

Hint 1: There is a normal subgroup N of order 77. Why?

Hint 2: What are the automorphisms of N?

Hint 3: N has exactly 4 automorphisms of order 5. Why?

Hint 4: By the N/C theorem there are exactly 5 ways for the generator of G/N to act on N by conjugation.

Hint 5: 4 of these are equivalent by changing the choice of generator of G/N. So, there are only two possibilities.

5.2.3. Suppose that P is a normal Sylow p-subgroup of a finite group G. Then show that P contains all elements of G of order p.

You prove this by contradiction. You assume it isn't true. Then G contains an element g of order p where $g \notin P$. Then you can proceed in two ways. You can show that gP has order p in the factor group G/P or you can show that $P\langle g\rangle$ is a p-group which is bigger than P. Either way you get a contradiction. It is a good idea to quote general facts in support of specific statements. For example "This step follows from the fact that the product of a subgroup and a normal subgroup is a subgroup."

5.2.4. Show that groups of order p^2q where $p < q$ are prime. Then show that G has a normal Sylow subgroup. (I'll send you hints later.)

Since q is a prime number, the congruence $p^2 \equiv 1 \mod q$ has only two solutions, namely $p \equiv \pm 1 \mod q$. (But there may be more solutions when the modulus is not prime. For example, there are 4 solutions to the equation $x^2 \equiv 1 \mod 8$.) In particular $p \geq q - 1$. What are p and q? After you figure out p and q you will see that we did this example in class.