4. Answers to Quiz 3b

Let $G = D_4 \oplus D_4$ where D_4 is the dihedral group of order 8.

(1) Find two nontrivial proper subgroups H, N of G so that N is normal and H is not normal in G. (*nontrivial* means more than one element and *proper* means not equal to G.)

There are many answers. But the bigger N is in this answer, the easier it will be to answer the other two questions. Some of you found the normal subgroup $N = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ which has index 4. This is normal since it is the product of two normal subgroups.

Many of you gave the answer:

$$N = Z(D_4) \oplus Z(D_4)$$

which has 4 elements. So, you had (or should have had) $|G/N| = 64/4 = 16$ elements in part 2 and 3.

There are lots of nonnormal subgroups, for example $H = \langle (s,e) \rangle = \{e, (s,e)\}$ is not normal where s is any reflection. This subgroup is not normal because $(r,e)H(r,e)^{-1} = \{e, (sr^2, e)\} \neq H$.

A lot of you were confused about the notation. The elements of the group G have parentheses around two letters: (a,b). Pointy brackets ($\langle\rangle$) are to denote subgroups generated by certain elements. For example here are some correct notations:

$$N = Z(D_4) \oplus Z(D_4) = \left\langle (e, r^2), (r^2, e) \right\rangle = \left\langle r^2 \right\rangle \oplus \left\langle r^2 \right\rangle = \{e, r^2\} \oplus \{e, r^2\}$$

A few of you got the wrong answer: $N = \langle (r,r) \rangle$. This is not a normal subgroup since it does not contain the conjugate $(s,e)(r,r)(s,e)^{-1} = (sr^3, r)$ of $(r,r) \in N$. However, I also thought it was normal at first ...

(2) List the elements of the factor group G/N.

If you had $N = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ you should have gotten $64/16 = 4$ elements in G/N. They are: $N, (e,s)N, (s,e)N, (s,s)N$. if you used $N = Z(D_4) \oplus Z(D_4)$ then you should have gotten $64/4 = 16$ cosets: $(a,b)N$ where $a, b = e, r, s, sr$.

(3) Find the order of each of the elements of G/N.

In both examples, the identity N has order 1 and the other elements all have order 2.