Functions are written $f : A \to B$, $a \mapsto f(a)$. This means A is the *domain* of f and B is the *range*. I spent a lot of time explaining that range is not the same as *image*. More later.

The *image* of f is the set of all values:

$$im(f) := \{f(a) \mid a \in A\}$$

The function f is *onto* or *surjective* if the image of f is equal to its range. Can you prove that the composition of surjective mappings is surjective?

The *Axiom of Choice* says that any surjective mapping has a right inverse.

The function f is *one-to-one* if any two distinct elements of A give different elements of B

$$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$$

(Thus "one-to-one" really means *two-to-two*.) Can you show that any composition of 1-1 mappings is 1-1? If A is nonempty, then f has a left inverse.

A function $f : A \to B$ which is both 1-1 and onto is called a *bijection*. When $A = B$ a bijection $f : A \to A$ is called a *permutation* of A. Any permutation has a unique (two-sided) inverse f^{-1}.

The set of permutation of A forms a group under composition. We will study this in Chapter 5.

Here is an example of a permutation of the set $\mathbb{R}$ of real numbers:

$$f(x) = 3 - 2x, \quad f^{-1}(x) = \frac{3 - x}{2}$$

1. Groups

The original concept of a group was that a group was the set of symmetries of some object or set. Around 1900 Artin and Noether invented "abstract groups" which are just sets with operations satisfying some axioms. The word "abstract" refers to the fact that there is no object of which the group is the set of symmetries of. (Sorry for the bad grammar.)

1.1. **dihedral group D_n.** So, following history, we first look at a concrete group: the group of symmetries of a square. A square has 8 symmetries: 4 rotations (by $0, \pi/2, \pi, 3\pi/2$) and 4 reflections. The book writes the rotations as: $R_0, R_{90}, R_{180}, R_{270}$ and the reflections as H, V, D, D'. Note: in algebra the main diagonal goes from upper left to lower right. (E.g., in a matrix the diagonal entries a_{11}, a_{22}, etc go that way.)

In order to multiply symmetries of a square we need to think of them (or at least write them) as permutations of the corners that we label $0, 1, 2, 3$ (starting at the upper left and going counterclockwise). Then R_{90} moves 0 to 1 and 1 to 2, etc.

$$R_{n\pi/2}(x) = x + n \qquad \mod 4$$

The four reflections are given by:

$$\begin{aligned} V(x) &= 3 - x \\ H(x) &= 1 - x \mod 4 \\ D(x) &= -x \mod 4 \\ D'(x) &= 2 - x \mod 4 \end{aligned}$$

With these equations we can compose:

$$HR_{90}(x) = H(x + 1) = 1 - (x + 1) = -x = D(x)$$

So, $HR_{90} = D$.

$$R_{90}H(x) = R_{90}(1 - x) = 2 - x = D'(x)$$

So, $R_{90}H = D' \neq HR_{90}$. This group is *not commutative.*

The group of symmetries of the square is called the *dihedral group* of order 8 and written D_4.

Definition 1.1. *The* dihedral group D_n *is defined to be the group of symmetries of the regular n-gon. This group has n rotations r_i and n reflections s_i (where $i = 0, \cdots, n-1$) which are given by permuting the vertices* $0, 1, \cdots, n-1$ *by the formulas:*

$$r_k(x) := x + k, \quad s_k(x) := k - x \mod n$$

Problem: Show that every rotation is the product (composition) of two reflections.

1.2. **cyclic groups $\mathbb{Z}_n$.** Some objects with rotational symmetry are "right handed" or "left handed" and are not the same as their mirror images, e.g., a propeller or screw. If the rotations $R_{k\pi/n}$ for $k = 0, 1, 2, \cdots, n-1$ are the only symmetries then the symmetry group is the *cyclic group* of order n. This is called $\mathbb{Z}_n$ because the composition rule is given by addition modulo n.

$$R_{i\pi/n}R_{j\pi/n} = R_{(i+j)\pi/n}$$