

9.3. internal direct products. A group is an internal direct product if it is a product of two subgroups. These subgroups are necessarily normal.

Definition 9.11. A group G is the internal direct product of H and K if H, K are normal subgroups of G so that

$$G = HK$$

and

$$H \cap K = \{e\}$$

The book writes $G = H \times K$ to indicate that G is an internal direct product.

The first equation $G = HK$ says that every $g \in G$ can be written as $g = hk$. The second statement, $H \cap K = \{e\}$ says that h, k are unique, i.e., if $h_1k_1 = h_2k_2$ then $h_1 = h_2$ and $k_1 = k_2$. The reason is that the equation $h_1k_1 = h_2k_2$ gives:

$$h_2^{-1}h_1 = k_2k_1^{-1}$$

This is an element of both H and K . So, it must be e . The uniqueness means that the elements $hk, h \in H, k \in K$ are all different. So $|G| = |HK| = |H| \cdot |K|$.

Theorem 9.12. $H \times K \cong H \oplus K$.

Here are two examples. The first is abelian. The second is not. Let $G = U(16)$. This has 8 elements. Let $H = \langle 3 \rangle = \{1, 3, 9, 11\}$ and $K = \langle 7 \rangle = \{1, 7\}$. We can see that $H \cap K = \{1\}$. This implies that HK has 8 elements and is therefore all of $U(16)$. So, the theorem says

$$U(16) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$$

Now take $G = GL(3, \mathbb{R})$ (the group of all 3×3 invertible real matrices with subgroups $H = SL(3, \mathbb{R}) = \{A \in GL(3, \mathbb{R}) \mid \det A = 1\}$ and let K be the group of all diagonal matrices

$$D(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$$

This group is isomorphic to the multiplicative group \mathbb{R}^* of nonzero real numbers. And K is central and thus normal in $GL(3, \mathbb{R})$. To see that $SL(3, \mathbb{R})$ is normal let $X \in SL(3, \mathbb{R})$ and $A \in GL(3, \mathbb{R})$. Then AXA^{-1} has determinant

$$\det(AXA^{-1}) = \det A \det X \det A^{-1} = \det A \det A^{-1} = 1.$$

So, $SL(3, \mathbb{R})$ contains the conjugates of all of its elements and is thus normal.

Since $\det D(x) = x^3$ is equal to 1 iff $x = 1$, we see that $H \cap K = \{I_3\}$. For any invertible matrix A we have $x = \det A \in \mathbb{R}^*$. This number has a cube root $y = x^{1/3}$ and we have $AD(1/y) \in SL(3, \mathbb{R})$. So,

$$A = (AD(1/y))D(y) \in HK$$

Therefore $GL(3, \mathbb{R}) = SL(3, \mathbb{R}) \times K \cong SL(3, \mathbb{R}) \oplus \mathbb{R}^*$.

In order to prove the theorem we need to show that the elements of H and K commute, i.e., that $hk = kh$ for all $h \in H$ and $k \in K$. This follows from the fact that the commutator $h^{-1}k^{-1}hk$ lies in the intersection $H \cap K$ and is therefore equal to e :

$$h^{-1}k^{-1}hk \in H(k^{-1}Hk) = HH = H$$

since H is normal and

$$h^{-1}k^{-1}hk \in (h^{-1}Kh)K = KK = K$$

since K is normal.

Now we can show that $H \times K \cong H \oplus K$. Let

$$\phi : H \oplus K \rightarrow G = H \times K$$

be given by $\phi(h, k) = hk$. This is onto since $G = HK$ and it is 1-1 since $H \cap K = \{e\}$. It is a homomorphism (OP) since h, k commute:

$$\phi(h_1, k_1)\phi(h_2, k_2) = h_1k_1h_2k_2 = h_1h_2k_1k_2 = \phi(h_1h_2, k_1k_2).$$