# MATH 30A: CHAPTER 25

## 25. Finite simple groups

A *simple group* is a nontrivial group $G$ which has no nontrivial proper normal subgroups. I.e, the only normal subgroups of $G$ are the trivial group $\{e\}$ and $G$ itself. (A subgroup $H \leq G$ is called *proper* if $H \neq G$ and it is called nontrivial if it has more than one element.)

**Theorem 25.1.** *An abelian group is simple if and only if it is cyclic of prime order.*

*Proof.* $\mathbb{Z}_p$ is simple by Lagrange: The number of elements in any subgroup $H$ must divide $p$ so $|H| = 1$ or $p$. Conversely, suppose that $G$ is a simple abelian group. Since all subgroups are normal, this means that the only subgroups of $G$ are $G$ and $\{e\}$. Then, first of all $G$ must be cyclic since any nontrivial element $g$ generates a cyclic subgroup $\langle g \rangle$ which must be equal to $G$ since $G$ is simple. The rest is by elimination: $\mathbb{Z}$ is not simple since $2\mathbb{Z}$ is a nontrivial subgroup. And $\mathbb{Z}_{nm}$ is not simple since it contains a cyclic subgroup of order $n$. That leaves only $\mathbb{Z}_p$. □

### 25.1. nonsimplicity.

On the first day I will talk about groups which are not simple. These theorems say "there is no simple group of order $n$."

#### 25.1.1. *using Sylow's third.*

**Problem 1**: Use Sylow's 3rd thm to show that there is no simple group of order 42. Hint: How many Sylow-7 subgroups does it have?

**Theorem 25.2.** *If $p$ divides the order of a finite group $G$ and $n = |G|$ has no factors congruent to 1 modulo $p$ then $G$ cannot be simple.*

*Proof.* By Sylow's 3rd, the index of the normalizer of a $p$-Sylow subgroup $P$ is congruent to 1 modulo $p$

$$|G : N(P)| \cong 1 \mod p$$

But this number is a divisor of $n$. So, it must be 1 and $P \triangleleft G$. □

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**Problem 2**: Show that a group of order $5481 = 3^3 7 \cdot 29$ is not simple. Hint: How many Sylow-3 subgroups does $G$ have? Then use the following lemma and ask yourself: Does $S_7$ contain an element of order 29?

**Lemma 25.3.** *If a group acts nontrivially on a set with $k$ elements, then there is a nontrivial homomorphism from $G$ to $S_k$.*

*Proof.* This is a rewording of the axiom:

$$g(hx) = (gh)x$$

$$\text{(action of)}(g) \circ \text{(action of)}(h) = \text{(action of)}(gh)$$

Therefore "action of" is a homomorphism from $G$ to the permutation group on $X$. □

25.1.2. *permuting cosets.* Suppose that $H$ is a subgroup of $G$ and $m = |G : H|$. Then $m$ is the number of left cosets $gH$ of $H$. The group $G$ acts on this set by left multiplication and we get a homomorphism

$$\phi : G \to S_m$$

What do we know about the kernel $K = \ker \phi$?

(1) $K \lhd G$ (all kernels are normal)
(2) $K \subseteq H$ (if $g \notin H$ then $gH \neq H$ so $g \notin K$)

If we want to do more we have to look closer at the definition of $K = \ker \phi$:

$$K = \{k \in G \mid kgH = gH \forall g \in G\}$$

The condition is satisfied if and only if $g^{-1}kg \in H$ and this is the same as $k \in gHg^{-1}$ for all $g \in G$. In other words $K$ is the intersection of all conjugates of $H$:

$$K = \ker \phi = \bigcap_{g \in G} gHg^{-1}$$

This proves the following:

**Theorem 25.4** (generalized Cayley theorem)**.** *Let $H$ be a subgroup of $G$ and consider the action of $G$ on the set of left cosets of $H$ by left multiplication. Then the kernel $K$ of this action is a normal subgroup of $G$ which is contained in $H$ and in every conjugate of $H$. Furthermore $K$ contains any other subgroup of $G$ which is contained in every conjugate of $H$.*

**Problem 3**: Show that any group of order $5103 = 3^6 7$ has a nontrivial normal 3-subgroup (a nontrivial subgroup $H$ whose order is a power of 3). Hint: $7! = 5040 < 5103$.

25.1.3. *permuting elements.* Instead of permuting subgroups, we can permute elements.

**Theorem 25.5.** *If $|G|$ is twice an odd number (and $|G| \neq 2$) then $G$ is not simple.*

*Proof.* $G$ acts on $X = G$ by multiplication and an element $g \in G$ of order 2 will give an odd permutation. The elements which give an ever permutation will form a subgroup of index 2 which is normal. □

This can be generalized to the following:

**Theorem 25.6.** *A simple group cannot have a nontrivial cyclic Sylow-2 subgroup.*