3. Math 30a, Fall 2009
Homework 3 Answers to odd problems

Due: Thursday, Sept 17, at noon (in class).

Homework counts 50% of your grade. Late homework will incur a penalty.

3.1. **Problems from section 2.** Page 26, numbers 7[1],

(Commutativity) The operation $a * b = a - b$ on $\mathbb{Z}$ is obviously not commutative, for example: $3 * 1 = 2$ is not the same ast $1 * 3 = -2$.

(Associativity) The operation is also not associative. For example:

$$3 * (2 * 1) = 3 * 1 = 2$$

is not equal to

$$(3 * 2) * 1 = 1 * 1 = 0$$

17. The operation $a * b = a - b$ is not a binary operation on the set $\mathbb{Z}^+$ since the result is not always in the set. For example $1 * 3 = -2$ is not in $\mathbb{Z}^+$.

27. If S has only one element then any equality statement involving elements of the set will be true. Therefore, commutativity and associativity hold.

3.2. **Problems from section 3.** Page 34, numbers 7. The mapping $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(x) = x^3$ is a bijection since every element has a unique real cube root. The proof is that there is an inverse function ψ given by cube root. However, fractional roots are only defined for positive real numbers (by the equation $a^b = e^{b \ln a}$ (and $e^x = 1 + x + x^2/2 + x^3/3! + \cdots$). So the cube root is given by the equation:

$$\psi(x) = \begin{cases} x^{1/3} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -|x|^{1/3} & \text{if } x < 0 \end{cases}$$

The homomorphism property for ϕ is easily verified:

$$\phi(xy) = (xy)^3 = xyxyxy = xxxyyy = x^3y^3 = \phi(x)\phi(y)$$

since multiplication of real numbers is commutative[2].

17. We have a bijection $\phi : \mathbb{Z} \to \mathbb{Z}$ ($\phi(n) = n + 1$) with inverse function $\phi^{-1}(x) = x - 1$.

[1]The answer to 2.7 in the back of the book is not a complete sentence! I object!

[2]If we had the definition of real numbers and the definition of what precisely is the product of such numbers then it would be easy to show that multiplication of real numbers is commutative. But this is a very long story which is the topic of another course.

a) We want ϕ to give an isomorphism $(\mathbb{Z}, \cdot) \to (\mathbb{Z}, *)$.
We want:

$$\phi(ab) = \phi(a) * \phi(b)$$

If we write $x = \phi(a), y = \phi(b)$ then this becomes:

$$x*y = \phi(\phi^{-1}(x)\phi^{-1}(y)) = \phi((x-1)(y-1)) = (x-1)(y-1)+1 = xy-x-y+2$$

The identity e must be

$$e = \phi(1) = 1 + 1 = 2.$$

This is the method for finding the answer. But it assumes that an answer exists. The above analysis proves that there is at most one solution. To prove that this is the answer we write:

Let $x * y := xy - x - y + 2 = (x - 1)(y - 1) + 1$ then

$$\phi(x) * \phi(y) = (x + 1) * (y + 1) = xy + 1 = \phi(xy)$$

The identity for $*$ is $e = 2$ since

$$x * 2 = (x - 1)(2 - 1) + 1 = (x - 1) + 1 = 2 * x$$

b) We want τ to be an isomorphism $(\mathbb{Z}, *) \to (\mathbb{Z}, \cdot)$ for some operation $*$. This time we do only the second part.

Let $*$ be given by

$$x * y = y * x = xy + x + y.$$

Then for any $x, y \in \mathbb{Z}$ we have

$$\phi(x*y) = (xy+x+y)+1 = xy+x+y+1 = (x+1)(y+1) = \phi(x)\phi(y).$$

In other words, ϕ satisfies the homomorphism property. This proves that $x * y = xy + x + y$ is a solution to the question. But, it does not preclude the existence of other solutions. (However, the solution is indeed unique. Why?)

Let $e = 0$. Then

$$0 * x = 0x + 0 + x = x$$

So, 0 is the identity of $(\mathbb{Z}, *)$. We know that the identity is unique when it exists.

Answer to 3.27: $\psi \circ \phi(x * y) =_1 \psi(\phi(x * y)) =_2 \psi(\phi(x) * \phi(y)) =_3 \psi(\phi(x)) * \psi(\phi(y)) =_4 (\psi \circ \phi)(x) * (\psi \circ \phi)(y)$ where (1) and (4) hold by definition of $\psi \circ \phi$, (2) holds since ϕ is an isomorphism and (3) holds since ψ is an isomorphism.

3.3. **Problems from section 4.** Page 45, numbers 17

(I should have said take $n = 2$ in number 17. The answer "yes" in the back of the book is just not a sufficient answer. However, the long explanation given for the $n \times n$ case below is more that I would expect.) Upper triangular 2×2 matrices are given by

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

The determinant ac is equal to 1 if and only if $c = 1/a$. So the set of 2×2 upper triangular matrices with determinant 1 is

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \text{ so that } a, b \in \mathbb{R}, a \neq 0 \right\}$$

(0) This set is closed:

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} = \begin{pmatrix} ax & ay + b/x \\ 0 & \frac{1}{ax} \end{pmatrix}$$

(1) Matrix multiplication is associative.

(2) S contains I_2 the 2×2 identity matrix which is the identity.

(3) The inverse of $\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$ is also upper triangular with determinant 1:

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & -b \\ 0 & a \end{pmatrix}$$

In the $n \times n$ case, it is helpful to write an upper triangular matrix as the product of a diagonal matrix and a *unipotent* matrix (upper triangular with 1's on the diagonal).

$$T = DU$$

The determinant of any unipotent matrix U is 1. The determinant of D is the product of the diagonal entries. For example

$$\begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & x/a & y/a \\ 0 & 1 & z/b \\ 0 & 0 & 1 \end{pmatrix}$$

with determinant $abc = 1$ which implies that the diagonal entries are all nonzero. Next write the unipotent matrix as the sum of an identity matrix and a strictly upper triangular matrix (with 0's on the diagonal)

$$U = I_n + X$$

for example

$$\begin{pmatrix} 1 & x/a & y/a \\ 0 & 1 & z/b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & x/a & y/a \\ 0 & 0 & z/b \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix X has the property that $X^n = 0$. In this example:

$$\begin{pmatrix} 0 & x/a & y/a \\ 0 & 0 & z/b \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & \frac{xz}{ab} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x/a & y/a \\ 0 & 0 & z/b \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So,

$$U^{-1} = (I + X)^{-1} = I - X + X^2 - X^3 + \cdots + (-1)^{n-1} X^{n-1}$$

This is unipotent since all powers of X are strictly upper triangular. Thus the inverse of an invertible upper triangular matrix is

$$T^{-1} = U^{-1} D^{-1} = (I - X + X^2 - X^3 + \cdots + (-1)^{n-1} X^{n-1}) D^{-1}$$

This is upper triangular since the product of the upper triangular matrix U^{-1} with the diagonal matrix D^{-1} is upper triangular. The determinant of T^{-1} is equal to $\det D^{-1} = (\det D)^{-1} = 1$. So, the inverse of a determinant one upper triangular matrix is upper triangular with determinant one.

There are other ways to do this problem using algebraic structures (sets with binary operations). We will see later in the course how messy computations become conceptual manipulations of sets, demonstrating the beauty of abstract algebra.

35. The statement is: $(ab)^2 = a^2 b^2$ for all $a, b \in G$ implies that $ab = ba$. This is really simple: We are given:

$$abab = aabb$$

We can cancel one a on the left by multiplying on the left with a^{-1} and we can cancel one b on the right by multiplying with b^{-1} to get

$$ba = ab$$

as desired.