

7. MATH 30A, FALL 2009  
HOMEWORK 7

New due date: Thursday, October 29. But don't let the homework pile up! HW7 still has 12 problems.

7.1. **Problems from section 13.** p. 133: # 3, 8, 20, 21, 22, 40, 52

7.2. **Problems from section 14.** p. 142 # 22. Here a new concept is defined in the problem! A group is called a **torsion group** if every element has finite order. Every finite group obviously has this property. So, the first question should be: Find an infinite group with this property. Many of you gave me such an example on your quiz!

**Problem 1** Show that the factor group  $\mathbb{Q}/\mathbb{Z}$  is a torsion group.

**22** Answer the questions in the book.

**40** Answer the question. Don't just say "We did part (a) in class" write out the answer, especially to part (b) which is slightly different.

**2** Answer the question using a general formula which applies to all questions 1-8.

**9** Answer the question using a general formula which applies to all questions 9-15.

### Solution of discussion problems

This is supposed to help do the homework.

34: If  $G$  is a group of order  $pq$  then show that every proper subgroup is cyclic.

First, you need to show that any group of prime order is cyclic. To see this suppose that  $G$  has order  $p$  and  $g \in G$  is not the identity. Then the order of  $g$  is a number which divides  $p$  and is not 1. So,  $o(g) = p$ . But this means that  $\langle g \rangle = G$ . So,  $G$  is cyclic.

Now suppose that  $|G| = pq$  and  $H$  is any proper subgroup of  $G$ . Then, by Lagrange,  $|H|$  must be either  $p, q$  or 1. So,  $H$  must be cyclic by the previous paragraph.

37: A group  $G$  has the property that (1) it has at least two elements and (2) it has no nontrivial proper subgroups (in other words, the trivial group  $\{e\}$  and the whole group  $G$  are the only subgroups of  $G$ ). Then show that  $G$  is (a) finite and (b) of prime order.

First note that (1) means that  $G$  has an element  $g$  which is not the identity. So, the cyclic subgroup  $H = \langle g \rangle$  is nontrivial. So, it must be the whole group. So,  $G = \langle g \rangle$  is cyclic. This means that  $G$  is isomorphic to either  $\mathbb{Z}$  or  $\mathbb{Z}_n$  for some integer  $n \geq 2$ . The properties (a) and (b) are structural properties shared by isomorphic groups. So we just have to show that (a)  $G$  cannot be equal to  $\mathbb{Z}$  and (b) If  $G = \mathbb{Z}_n$  then  $n$  is prime. Both of these proofs are by contradiction.

(a) Suppose that  $G = \mathbb{Z}$ . Then  $2\mathbb{Z}$  is a nontrivial proper subgroup and we have a contradiction.

(b) Suppose that  $G = \mathbb{Z}_n$  and  $n = pq$  where  $p$  is prime and  $q > 1$ . Then  $g^p$  has order  $q < pq = n$ . So  $\langle g^p \rangle$  is a nontrivial proper subgroup of  $G$ . This is a contradiction. So the factorization of  $n$  is impossible and we conclude that  $n$  is prime.