8. Math 30a, Fall 2009
Homework 8

New due date: Monday, Nov 2.

8.1. **Problems from section 16.** p. 159 # 1,2,3,11,12,

8.2. **Problems from section 17.** p. 164 # 4,5,7 But you need to say how the formula is used: Which formula is being used and what are the numbers being plugged into the formula. (I will do some examples in class.)

Answer to # 6 (modified): Each of the 8 corners of a cube is to be tipped with one of four colors, each of which may be used on any number of corners. Find the number of distinguishable markings. Use the hint: the group of rotations of the cube has 24 elements consisting of the identity, 9 which leave a pair of opposite faces invariant, 8 which leave a pair of opposite vertices invariant and 6 leaving a pair of opposite edges invariant. (*Invariant* means staying in the same place but possibly rotated in that place.)

The answer is:

$$\frac{1}{24}\left(8^4 + 3\cdot 4^4 + 6\cdot 2^4 + 8\cdot 4^4 + 6\cdot 4^4\right) = 356$$

This uses Burnside's formula:

$$r\cdot|G| = \sum_{g\in G}|X_g|$$

Where r is the answer to the question. G is the rotation group of the cube and $|G| = 24$ is given. X_g is the set of coloring of the cube which are fixed by the rotation g.

(1) For $g = e$ all patterns are fixed. So, $X_e = X$ which has 8^4 elements since each of the 8 corners has 4 possible colors.
(2) For g one of the 9 rotations which fix two opposite faces, there are 6 which are 90° rotations and 3 which are 180° rotations. The 3 which are 180° rotations have 4^4 possible colorings invariant under g since the 4 corners on the front must be the same color as the four in the back. This gives $|X_g| = 4^4$ three times. The 6 which are 90° rotations have 2^4 colorings giving $6\cdot 2^4$ in the sum.
(3) For g one of the 8 rotations which fix two opposite corners, these are 120° rotations which permute the other 6 corners in two 3-cycles. So, there are 4^4 ways to color this in a way fixed by the rotation. This gives $8\cdot 4^4$ in the sum.

(4) For g one of the 6 rotations which fix opposite edges, there are again 4^4 colorings fixed by g since the colors of the vertices on the front face determine the colors on the back face (if the fixed edges are pointing away from you). This gives the last term $6 \cdot 4^4$ in the sum.